PhD defense

Optimal curves and mappings valued in the Wasserstein space

Hugo Lavenant – under the supervision of Filippo Santambrogio

May 24th, 2019

The Wasserstein[1] space

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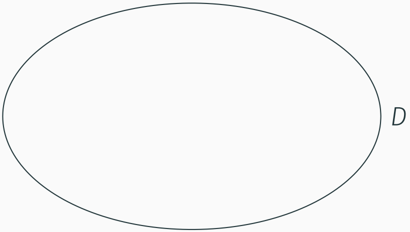

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The **Wasserstein space** is the space $\mathcal{P}(D)$ endowed with the Wasserstein distance.

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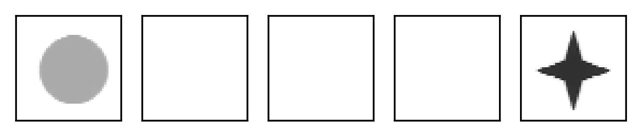
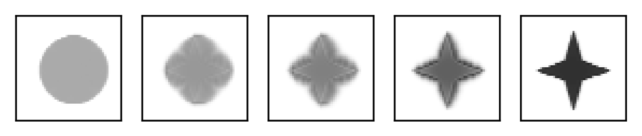
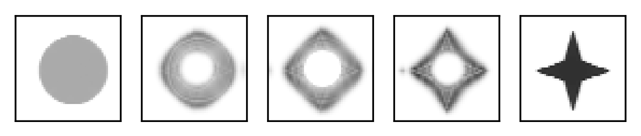
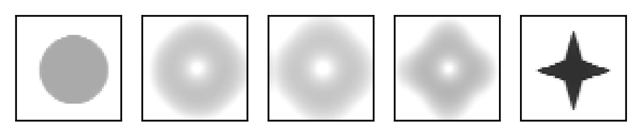
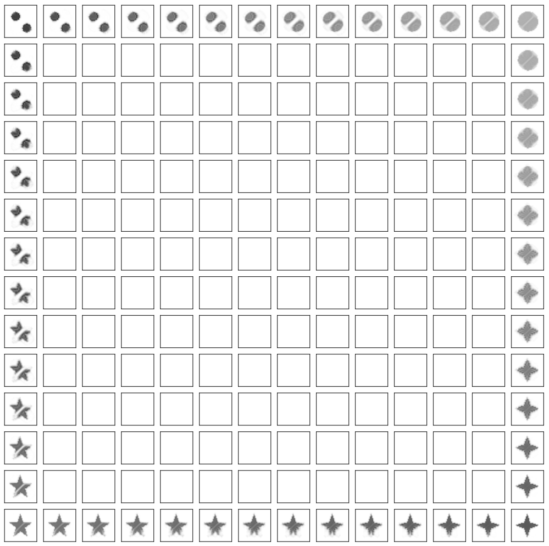
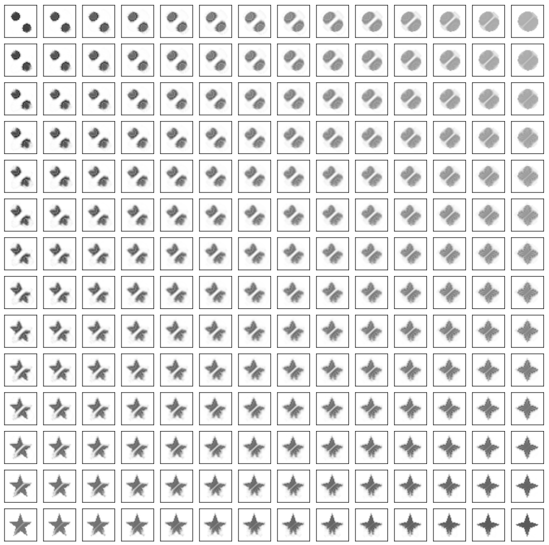










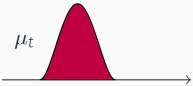


In this presentation

1. A quick introduction to the Wasserstein space
2. Optimal density evolution with congestion
3. Harmonic mappings valued in the Wasserstein space

1. A quick introduction to the Wasserstein space

The metric tensor in the Wasserstein space



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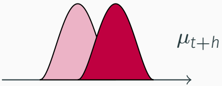

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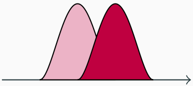
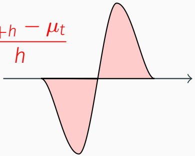


Vertical derivative

$$\frac{\mu_{t+h} - \mu_t}{h}$$



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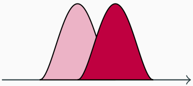

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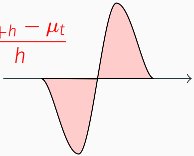

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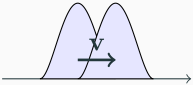

A particle located at x moves to $x + h\mathbf{v}(x)$

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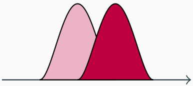

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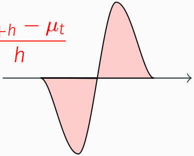

$$\boxed{-\partial_t \mu = \nabla \cdot (\mu \mathbf{v})}$$

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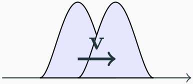

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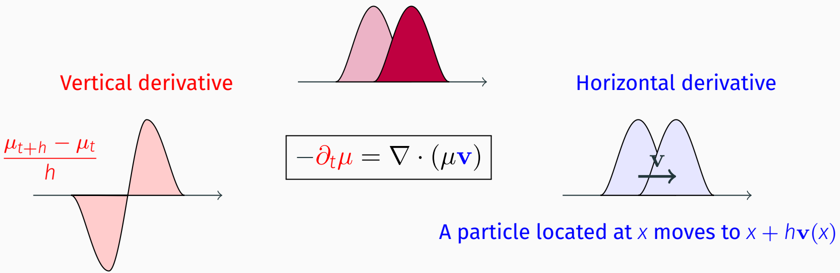

- Quadratic Optimal Transport: the square of the norm of the speed is

$$\min_{\mathbf{v}: D \to \mathbb{R}^d} \left\{ \int_D |\mathbf{v}(x)|^2 \, \mu(\mathrm{d}x) \ : \ \nabla \cdot (\mu \mathbf{v}) = -\partial_t \mu \right\}.$$

Action and geodesics

If $\mu : [0,1] \to \mathcal{P}(D)$ is given, its **action** is

$$\mathcal{A}(\mu) := \min_{\mathbf{v}} \left\{ \frac{1}{2} \int_0^1 \int_D |\mathbf{v}_t|^2 \, \mathrm{d}\mu_t \, \mathrm{d}t \ : \ \partial_t \mu_t + \nabla \cdot (\mu_t \mathbf{v}_t) = 0 \right\}.$$

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The Wasserstein distance W_2 is

$$\frac{1}{2} W_2^2(\rho, \nu) = \min_{\mu} \left\{ \mathcal{A}(\mu) \ : \ \mu_0 = \rho, \ \mu_1 = \nu \right\},$$

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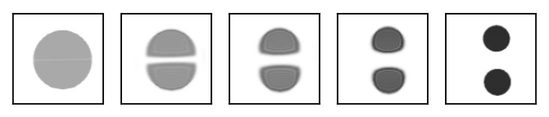
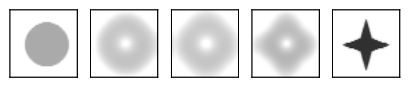


and the minimizers are the constant-speed geodesics.

2. Optimal density evolution with congestion



What do we minimize?

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Instanciation of a **Mean Field Game** of first order with local coupling.

Soft congestion

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Older result: use of a maximum principle by Lions to get L^{∞}.

Used to infer the **Lagrangian** interpretation of Mean Field Games.

Imply regularity of the value function (Cardaliaguet, Graber, Porretta, Tonon).

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If $m > 1$, with $\beta > 1$ such that $H^1(D) \hookrightarrow L^{2\beta}(D)$,

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Integration with respect to time and **Moser iterations**.

Need for a discretization of the temporal axis for a rigorous proof. □

Hard congestion

$$\min_{\mu}\left[\mathcal{A}(\mu)+\int_0^1\int_D V(x)\mu_t(x)\,\mathrm{d}x\,\mathrm{d}t+\int_D \Psi(x)\mu_1(x)\mathrm{d}x\right]$$

with μ_0 given and the constraint $\mu \leqslant 1$.

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Pressure $p \geqslant 0$ to enforce the constraint $\mu \leqslant 1$ (Cardaliaguet, Mészáros, Santambrogio).

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First approach: approximation by soft congestion.

Ultimately: estimate $\Delta(p + V) \geqslant 0$ on $\{p > 0\}$.

Variational formulation of the incompressible Euler equations

Least action principle: unknown Q law of a random curve $\mu : [0,1] \to \mathcal{P}(D)$.

$$\min_{Q} \left\{ \mathbb{E}_Q[\mathcal{A}(\mu)] \ : \ \forall t, \ \mathbb{E}_Q[\mu_t] = \mathcal{L}_D \right\},$$

with joint law at time $t \in \{0,1\}$ fixed.

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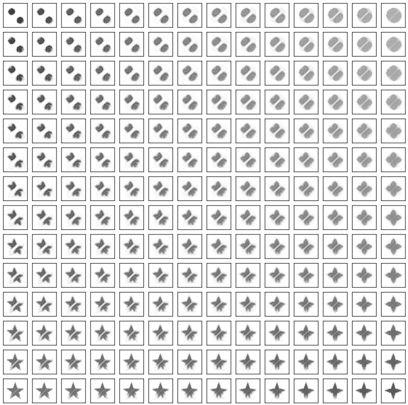
Similar discretization to make rigorous a formal computation of Brenier. □

Later simpler proof by Baradat and Monsaingeon.

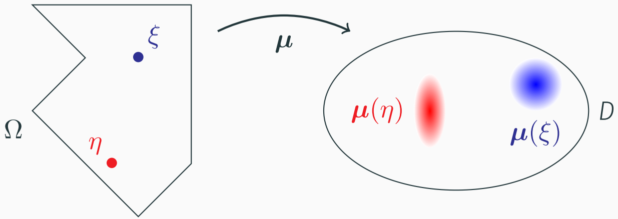
3. Harmonic mappings valued in the Wasserstein space



Measure-valued mappings

Ω bounded set of $\mathbb{R}^n$ with Lipschitz boundary

We study $\boldsymbol{\mu} : \Omega \to \mathcal{P}(D)$.



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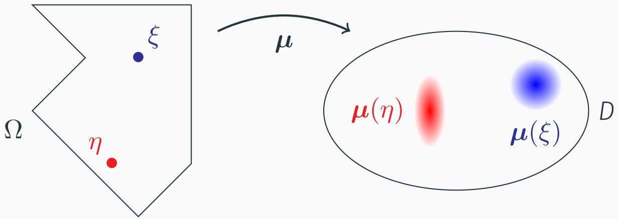

Definition of $\mathrm{Dir}(\boldsymbol{\mu}) = \dfrac{1}{2}\displaystyle\int_\Omega |\nabla \boldsymbol{\mu}|^2$ the **Dirichlet energy** generalizing $\mathcal{A}$.

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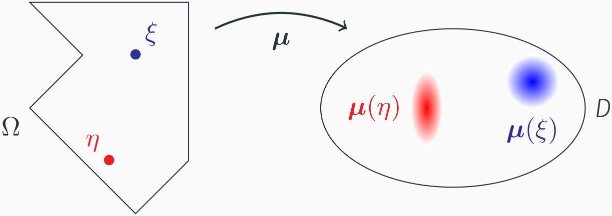

Definition of $\mathrm{Dir}(\boldsymbol{\mu}) = \dfrac{1}{2}\displaystyle\int_\Omega |\nabla \boldsymbol{\mu}|^2$ the **Dirichlet energy** generalizing $\mathcal{A}$.

Minimizers of Dir are called harmonic mappings (valued in the Wasserstein space).

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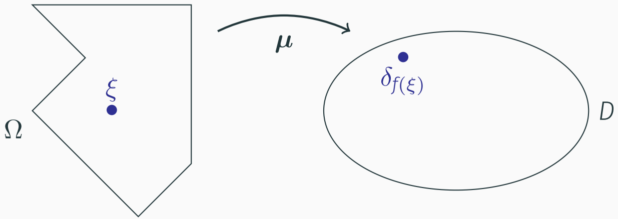
Ω bounded set of $\mathbb{R}^n$ with Lipschitz boundary (before $\Omega = [0,1] \subset \mathbb{R}$).

We study $\boldsymbol{\mu} : \Omega \to \mathcal{P}(D)$.



Definition of $\mathrm{Dir}(\boldsymbol{\mu}) = \dfrac{1}{2}\displaystyle\int_\Omega |\nabla \boldsymbol{\mu}|^2$ the **Dirichlet energy** generalizing $\mathcal{A}$.

Minimizers of Dir are called harmonic mappings (valued in the Wasserstein space).

If $f : \Omega \to D$ and $\boldsymbol{\mu}(\xi) := \delta_{f(\xi)}$ then $\mathrm{Dir}(\boldsymbol{\mu}) = \dfrac{1}{2}\displaystyle\int_\Omega |\nabla f|^2$.

The Dirichlet energy

Definition (Brenier (2003))

If $\boldsymbol{\mu} : \Omega \to \mathcal{P}(D)$ is given,

$$\mathrm{Dir}(\boldsymbol{\mu}) := \min_{\mathbf{v}} \left\{ \frac{1}{2} \int_\Omega \int_D |\mathbf{v}|^2 \mathrm{d}\boldsymbol{\mu} \ : \ \nabla_\Omega \boldsymbol{\mu} + \nabla_D \cdot (\boldsymbol{\mu}\mathbf{v}) = 0 \right\},$$

where $\mathbf{v} : \Omega \times D \to \mathbb{R}^{nd}$.

If $\Omega = [0, 1]$ it coincides with $\mathcal{A}$.

Equivalence with a metric definition

$$\frac{W_2^2(\boldsymbol{\mu}(\xi), \boldsymbol{\mu}(\eta))}{\varepsilon^2}$$

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$$\frac{1}{\varepsilon^n} \int_\Omega \frac{W_2^2(\boldsymbol{\mu}(\xi), \boldsymbol{\mu}(\eta))}{\varepsilon^2} \mathbb{1}_{|\xi - \eta| \leqslant \varepsilon} \, \mathrm{d}\eta$$

Equivalence with a metric definition

$$\mathrm{Dir}_\varepsilon(\boldsymbol{\mu}) := \frac{C_n}{2} \int_\Omega \frac{1}{\varepsilon^n} \int_\Omega \frac{W_2^2(\boldsymbol{\mu}(\xi), \boldsymbol{\mu}(\eta))}{\varepsilon^2} \mathbb{1}_{|\xi - \eta| \leqslant \varepsilon} \, \mathrm{d}\eta \, \mathrm{d}\xi$$

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Theorem

There holds

$$\lim_{\varepsilon \to 0} \mathrm{Dir}_\varepsilon = \mathrm{Dir},$$

and the convergence holds pointwisely and in the sense of Γ-convergence along the sequence $\varepsilon_m = 2^{-m}$.

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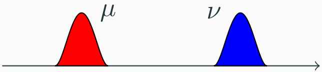
Cannot apply the whole theory of Korevaar, Schoen and Jost because the Wasserstein space is **not** a **Non Positively Curved** (NPC) space.

The space $\{\boldsymbol{\mu} \ : \ \mathrm{Dir}(\boldsymbol{\mu}) < +\infty\}$ coincides with $H^1(\Omega, \mathcal{P}(D))$ for the standard definitions of Sobolev spaces in metric spaces (Reshetnyak, Hajłasz).

Curvature and convexity

If $\mu, \nu \in \mathcal{P}(D)$, two ways to interpolate.



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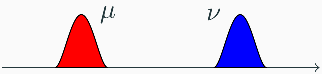

The **displacement** interpolation

- Midpoint of the geodesic in the Wasserstein space.
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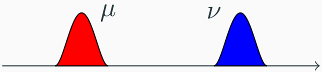

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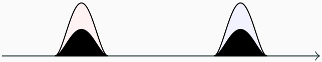

- The Wasserstein distance square W_2^2 and the Dirichlet energy are convex.
- Tools from convex analysis.

The Dirichlet problem

The Dirichlet problem

We choose $\boldsymbol{\mu}_b : \partial\Omega \to \mathcal{P}(D)$ the boundary data.

Definition

The Dirichlet problem is

$$\min_{\boldsymbol{\mu}} \left\{ \mathrm{Dir}(\boldsymbol{\mu}) \ : \ \boldsymbol{\mu} = \boldsymbol{\mu}_b \text{ on } \partial\Omega \right\}.$$

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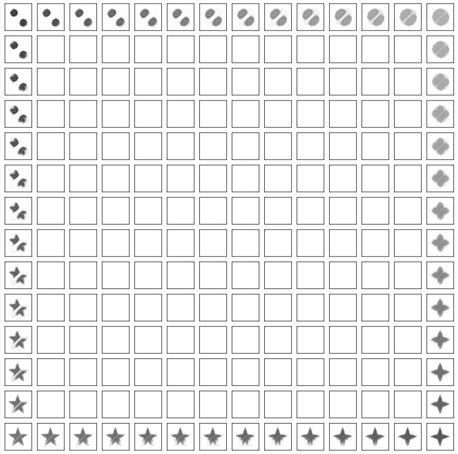
Theorem

Assume $\boldsymbol{\mu}_b : \partial\Omega \to (\mathcal{P}(D), W_2)$ is a Lipschitz mapping. Then there exists at least one solution to the Dirichlet problem.

Tool: extension theorem for Lipschitz mappings valued in $(\mathcal{P}(D), W_2)$. □

Uniqueness is an open question.

Numerics: example



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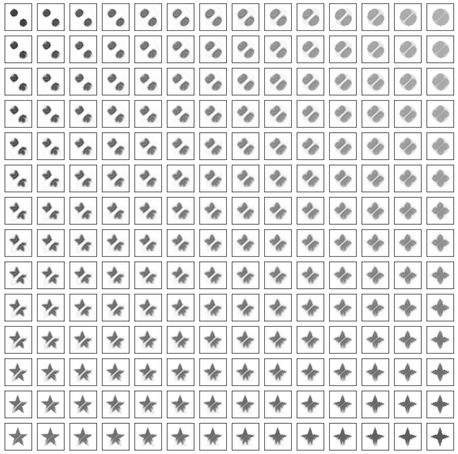

Numerics: adaptation of Benamou–Brenier

The Dirichlet problem is a convex optimization problem.

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Unknowns ($\mathbf{E} = \mu \mathbf{v}$ is the momentum):

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In practice: finite-dimensional “approximation” with two convex optimization problems in duality, then **ADMM**.

Maximum principle

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Some functionals $F : \mathcal{P}(D) \to \mathbb{R} \cup \{+\infty\}$ are convex along geodesics, e.g.

$$\mu \to \int_D \mu(x) \ln(\mu(x)) \, \mathrm{d}x.$$

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Take $F : \mathcal{P}(D) \to \mathbb{R} \cup \{+\infty\}$ convex along generalized geodesics (and few additional regularity property) and a boundary condition $\boldsymbol{\mu}_b : \partial\Omega \to \mathcal{P}(D)$ such that $\sup_{\partial\Omega}(F \circ \boldsymbol{\mu}_b) < +\infty$.

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Then there exists at least one solution $\boldsymbol{\mu}$ of the Dirichlet problem with boundary conditions $\boldsymbol{\mu}_b$ such that

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Already known for harmonic mappings valued in Riemannian manifolds (Ishihara) and Non Positively Curved spaces (Sturm).

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First replace Dir by the approximate Dirichlet energies Dir_ε.

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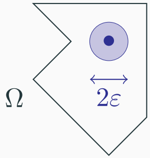
If $\boldsymbol{\mu}_\varepsilon$ minimizes Dir_ε, then for a.e. $\xi \in \Omega$, the measure $\boldsymbol{\mu}_\varepsilon(\xi)$ is a (Wasserstein) barycenter of the $\boldsymbol{\mu}_\varepsilon(\eta)$ for $\eta \in B(\xi, \varepsilon)$.



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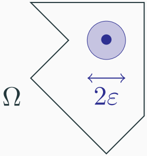

Jensen inequality for Wasserstein barycenters (Agueh, Carlier):

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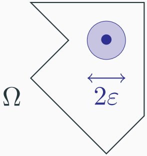

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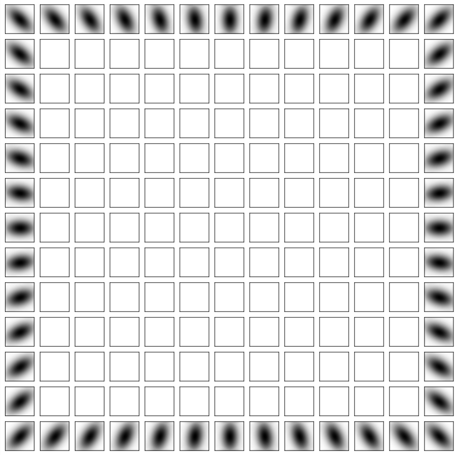
Then limit $\varepsilon \to 0$ to get subharmonicity. □

A special case

Family of "elliptically contoured distributions" $\mathcal{P}_{ec}(D)$, think Gaussians measures.



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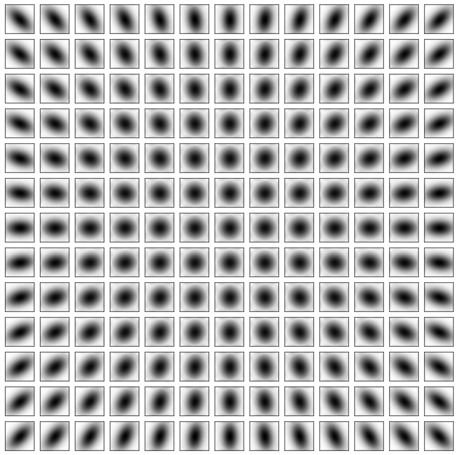

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Theorem

Let $\boldsymbol{\mu}_b : \partial\Omega \to \mathcal{P}_{ec}(D)$ *Lipschitz such that* $\boldsymbol{\mu}_b(\xi)$ *is not singular for every* $\xi \in \partial\Omega$.

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Theorem

Let $\boldsymbol{\mu}_b : \partial\Omega \to \mathcal{P}_{ec}(D)$ Lipschitz such that $\boldsymbol{\mu}_b(\xi)$ is not singular for every $\xi \in \partial\Omega$.

*Then there exists a **unique** solution to the Dirichlet problem, it is valued in $\mathcal{P}_{ec}(D)$ and it is **smooth**.*

Open questions and perspectives

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Optimal density evolution with congestion:

- Continuity of μ? Even more?

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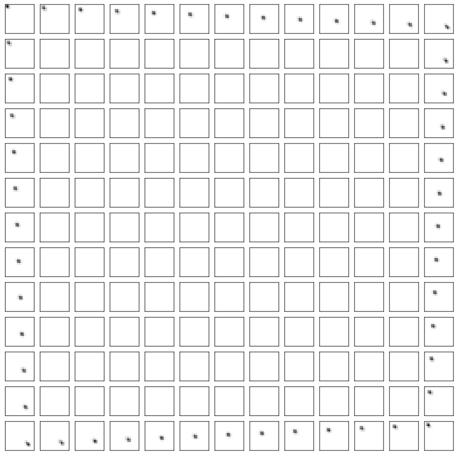
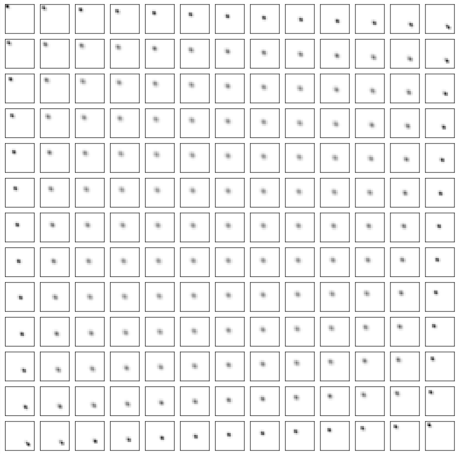
Thank you for your attention

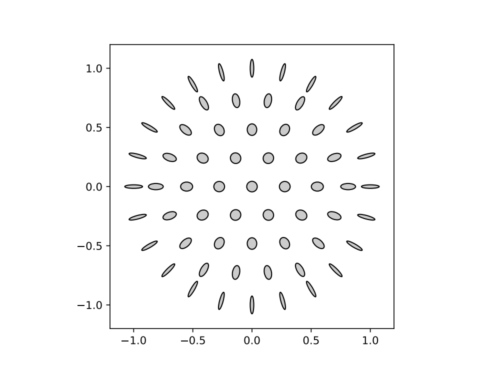
Appendix: Dirac masses

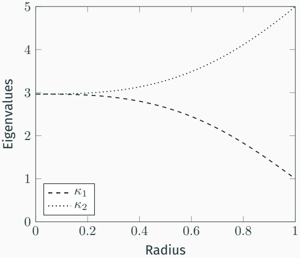


Appendix: Dirac masses

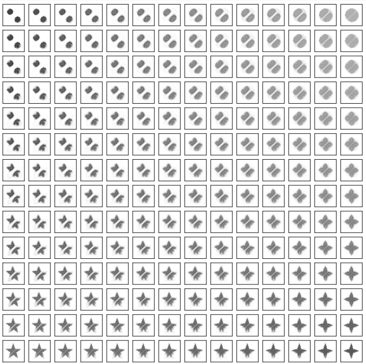


Appendix: an explicit example

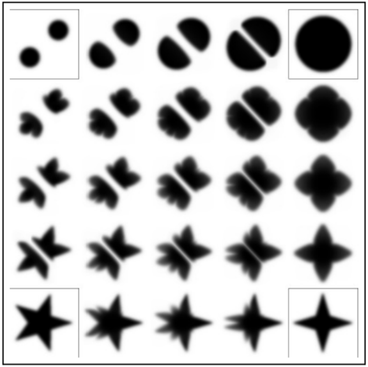




Appendix: harmonic and barycentric interpolations



Harmonic interpolation



Barycentric interpolation
(Solomon *et al.*, 2015)