Lifting functionals defined on maps to measure-valued maps via optimal transport

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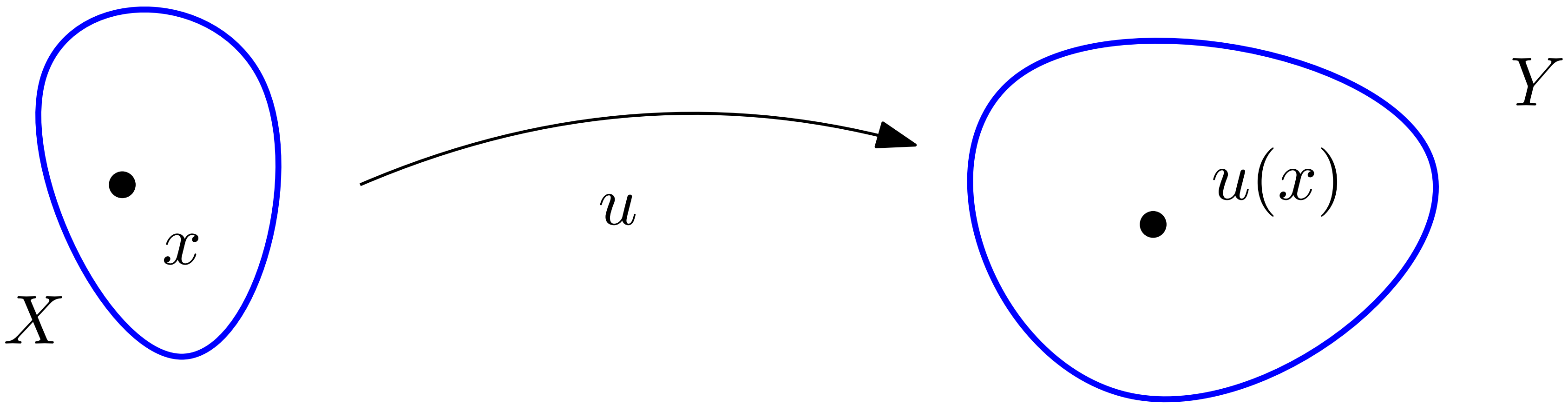
Bocconi University

MokaMeeting, of the INRIA MokaPlan team

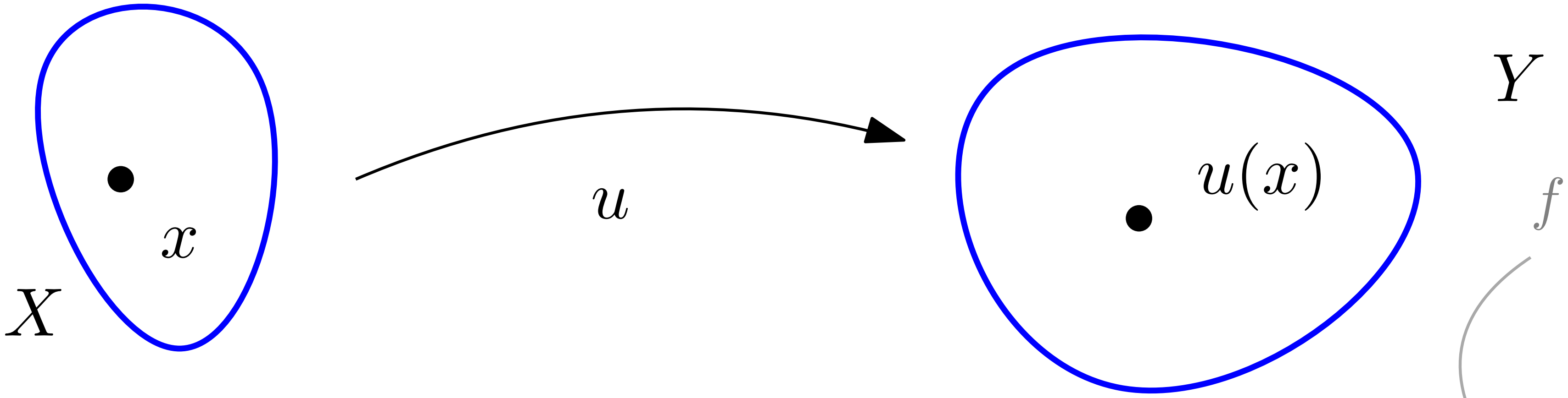
Paris (France), November 8, 2023

Lifting of functionals defined on maps



Given $\boxed{u \to E(u) \in [0, +\infty]}$

Lifting of functionals defined on maps



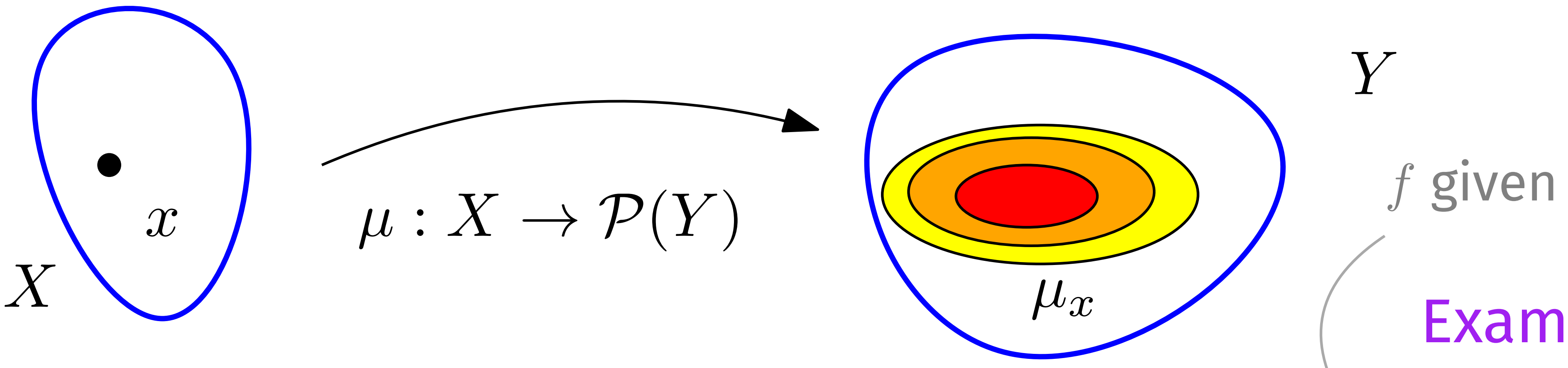
f given

Example:

$$E(u) = \int_X f(x, u(x))\, \mathrm{d}x$$

Given $\boxed{u \to E(u) \in [0, +\infty]}$

Lifting of functionals defined on maps



Example:

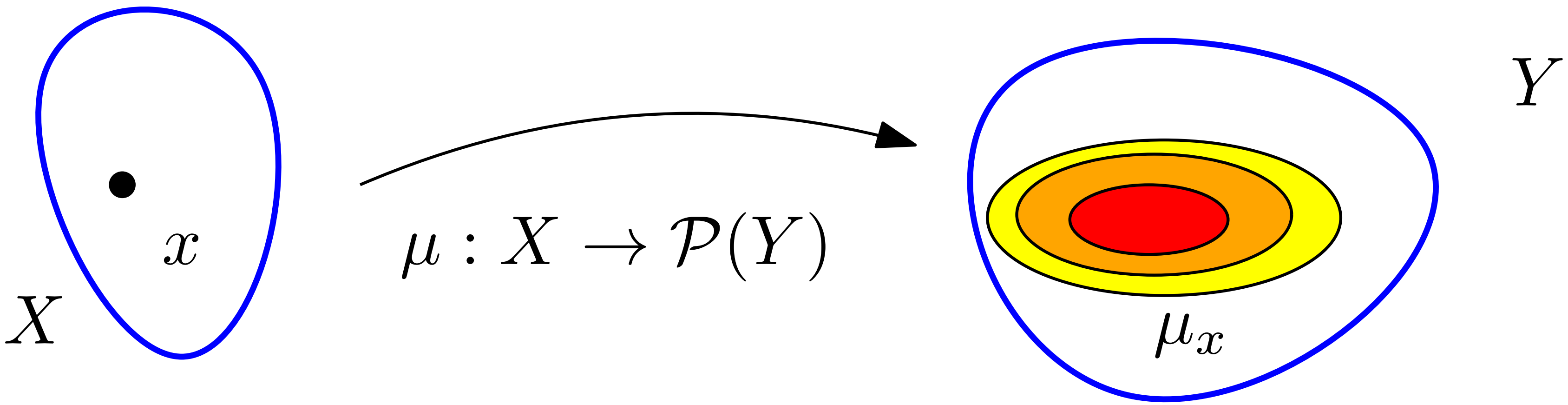
Given $\boxed{u \to E(u) \in [0, +\infty]}$

$$E(u) = \int_X f(x, u(x))\, \mathrm{d}x$$

Looking for $\boxed{\mu \to \mathcal{T}_E(\mu) \in [0, +\infty]}$

$$\mathcal{T}_E(\mu) = \int_X \left(\int_Y f(x, y)\, \mathrm{d}\mu_x(y) \right) \mathrm{d}x$$

Lifting of functionals defined on maps



Example:

Given

$$\boxed{u \to E(u) \in [0, +\infty]}$$

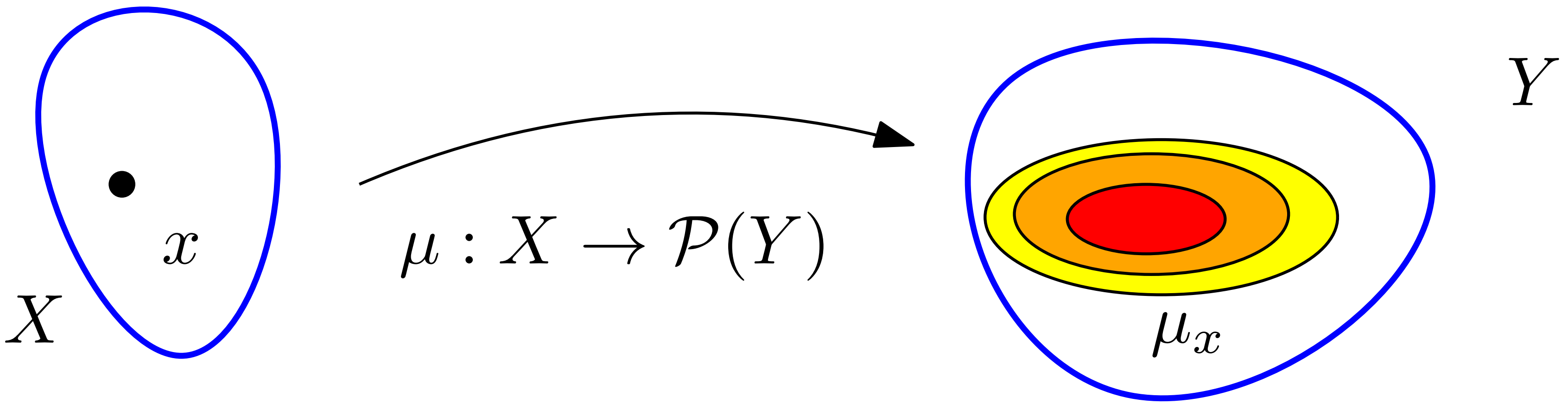
$$E(u) = \frac{1}{2} \int_X |\nabla u(x)|^2 \, \mathrm{d}x$$

Looking for

$$\boxed{\mu \to \mathcal{T}_E(\mu) \in [0, +\infty]}$$

???

Lifting of functionals defined on maps



Example:

Given $\boxed{u \to E(u) \in [0, +\infty]}$

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???

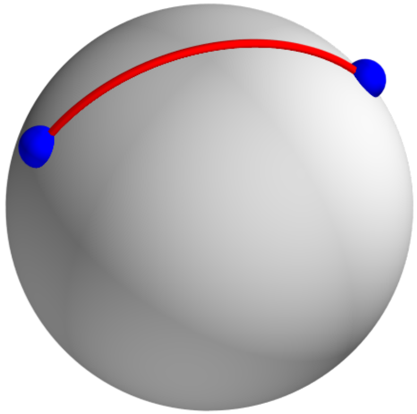
Today

- “Lagrangian” answer $\mathcal{T}_E$
- “Eulerian” answer $\mathcal{T}_{E,\mathrm{Eul}}$

Lifting of the action

$$X = [0,1], \qquad E(u) = \frac{1}{2}\int_0^1 |\dot{u}_t|^2 \,\mathrm{d}t$$

E minimized for (constant speed) geodesics



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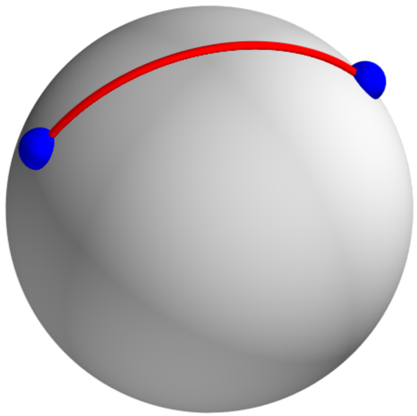
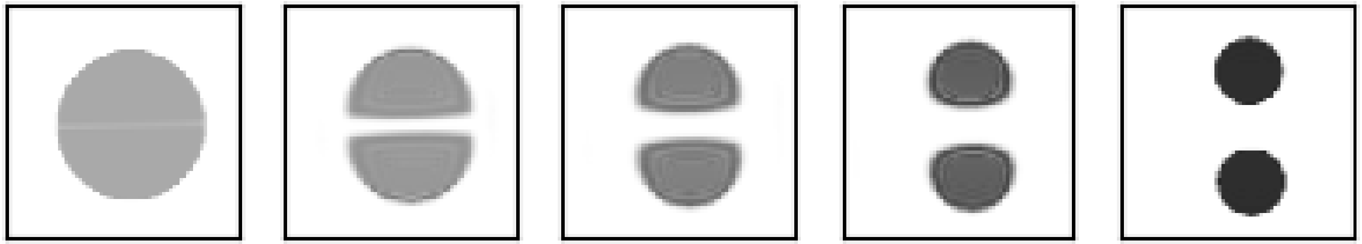
Minimizers of $\mathcal{T}_E$ are geodesics (for an optimal transport geometry)

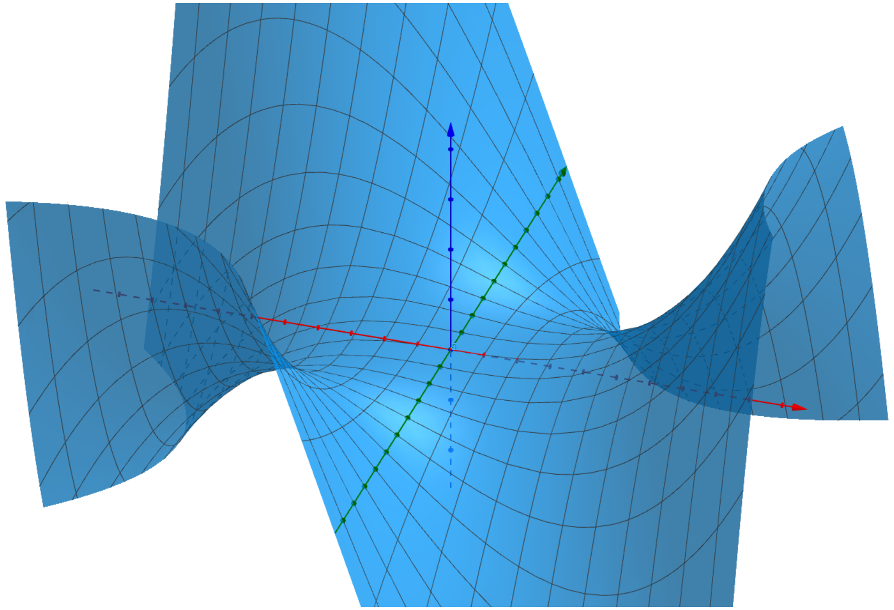




Lifting of the Dirichlet energy

$$E(u) = \int |\nabla u(x)|^2 \, \mathrm{d}x$$

E minimized for harmonic maps



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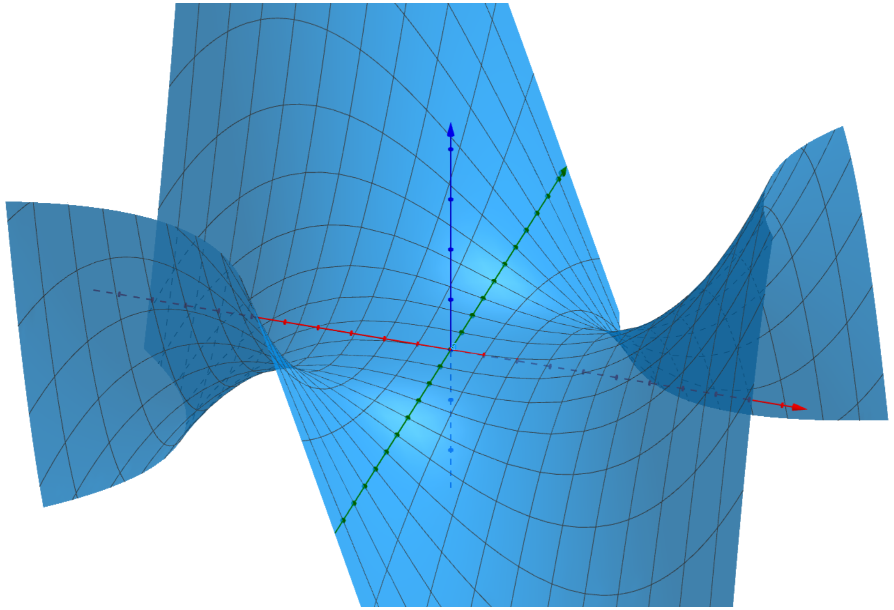
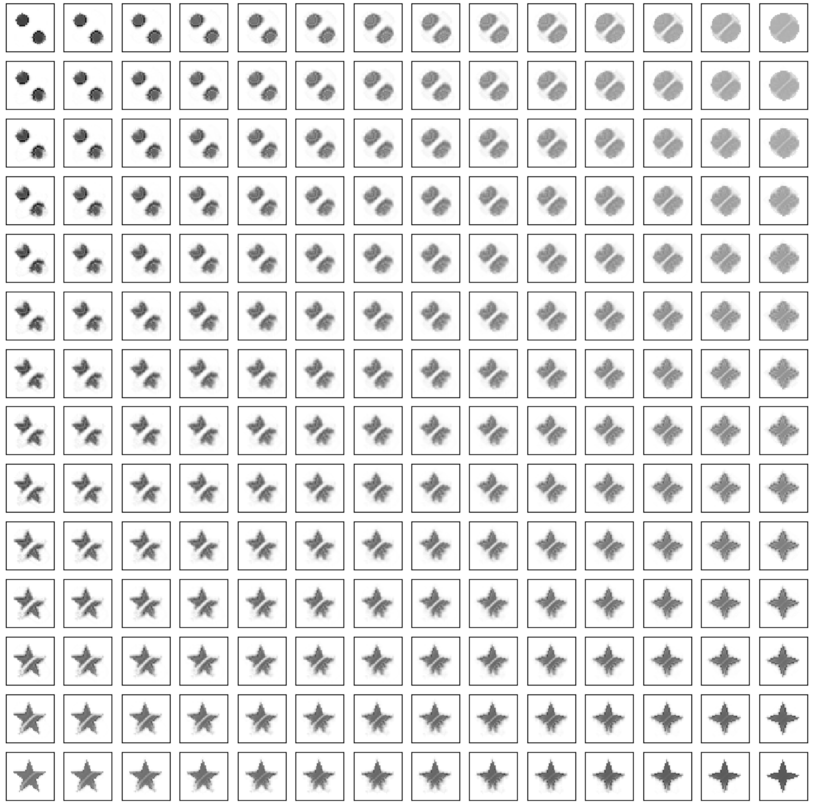


Minimizers of the **Eulerian** lifting of the Dirichlet energy are **harmonic measure-valued maps**.



Brenier (2003). Extended Monge-Kantorovich theory.
Solomon, Guibas and Butscher (2013). Dirichlet energy for analysis and synthesisof soft maps.
Lavenant (2019). Harmonic mappings valued in the Wasserstein space.

Why? map denoising, in imaging

minimize $$E(u) = \int W(\nabla u(x))\,\mathrm{d}x + \int f(x, u(x))\,\mathrm{d}x$$

Vogt, Haase, Bednarski and Lellmann (2020). On the connection between dynamical optimal transport and functional lifting.

Why? map denoising, in imaging

data fitting, like
$f(x, u(x)) = |u(x) - \overline{u}(x)|^2$

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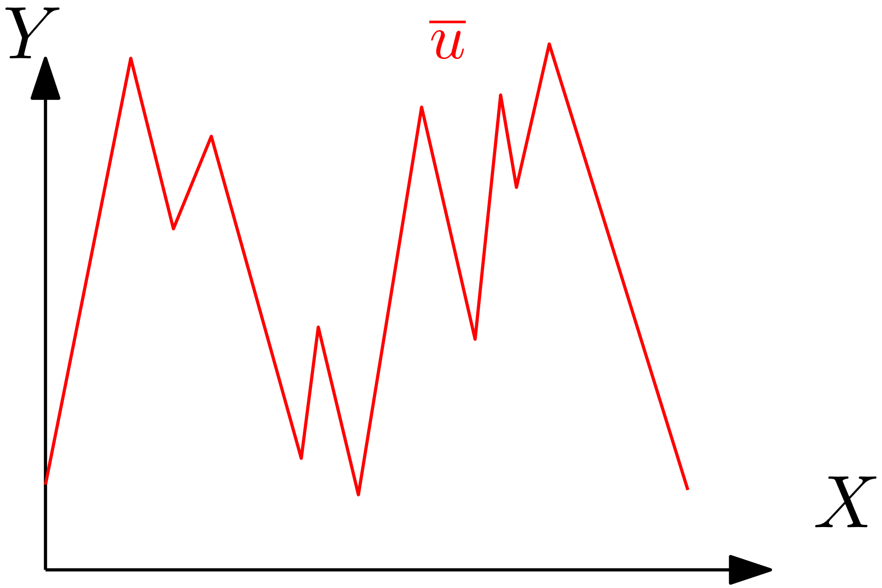
Regularization



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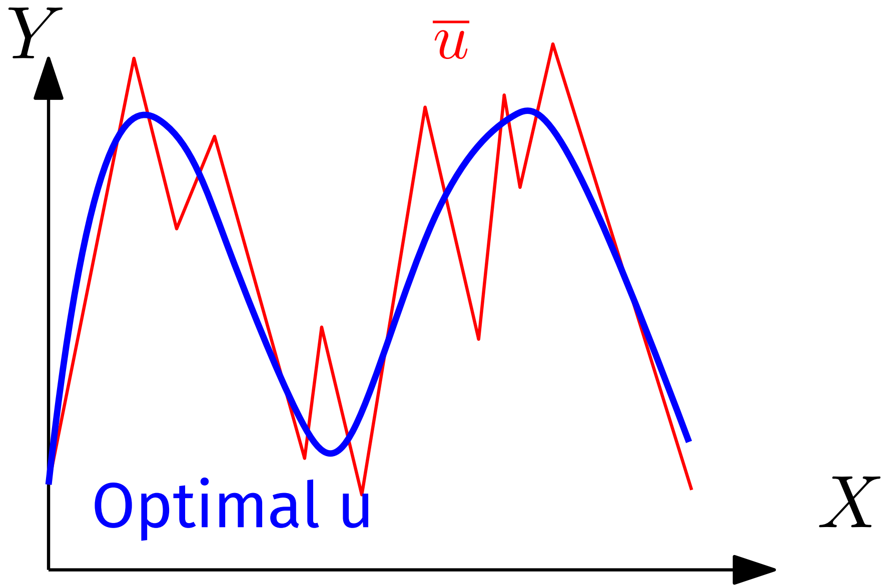

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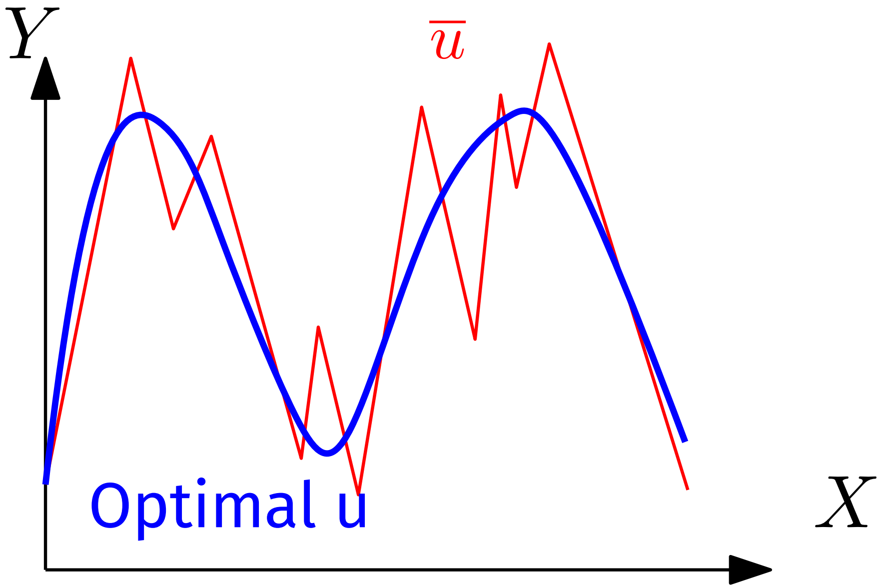
Regularization

Codomain of u manifold, or f non convex → **convexification.**



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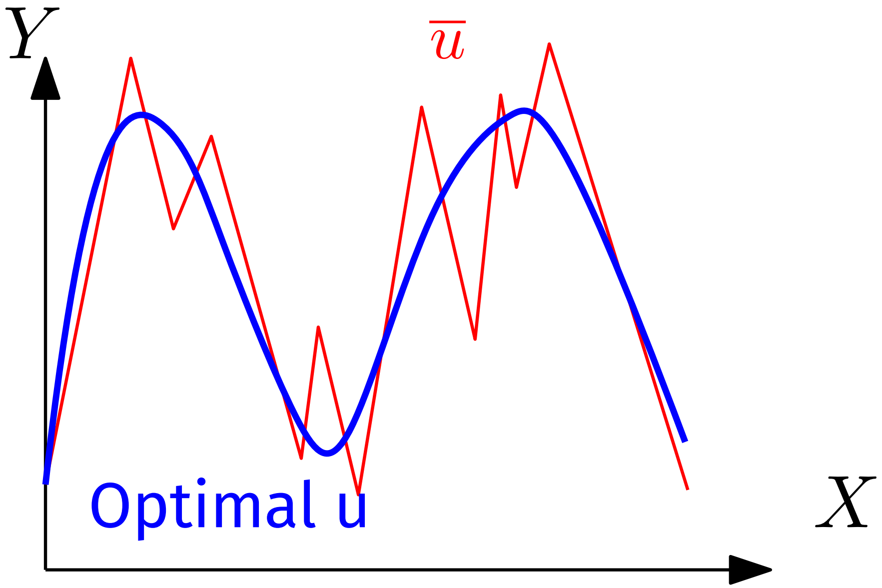
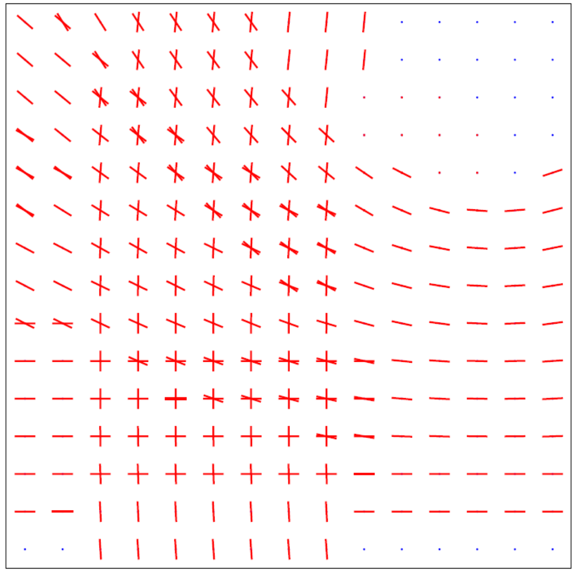




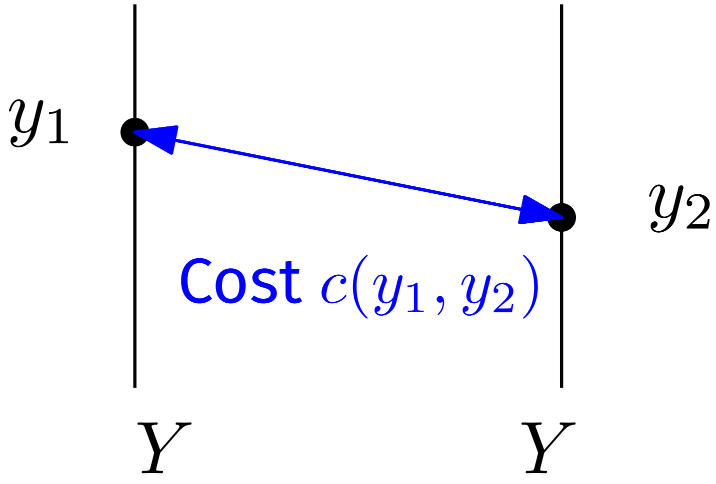
Data really measure-valued:

Magnetic Resonance Imaging: distributions of directions, in $\mathcal{P}(\mathbb{S}^2)$

Vogt, Haase, Bednarski and Lellmann (2020). On the connection between dynamical optimal transport and functional lifting.
Vogt and Lellmann (2018). Measure-valued variational models with applications todiffusion-weighted imaging.

The link with optimal transport

Simpler question: $c : Y \times Y \to [0, +\infty]$



Savaré and Sodini (2022). A simple relaxation approach to duality for Optimal Transport problems in completely regular spaces.

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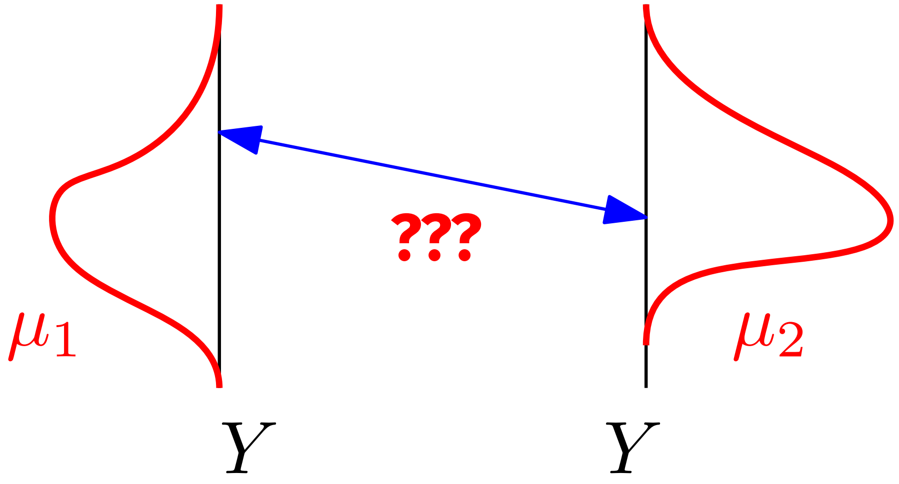

Question: how to extend c into $\mathcal{T}_c : \mathcal{P}(Y) \times \mathcal{P}(Y) \to [0, +\infty]$

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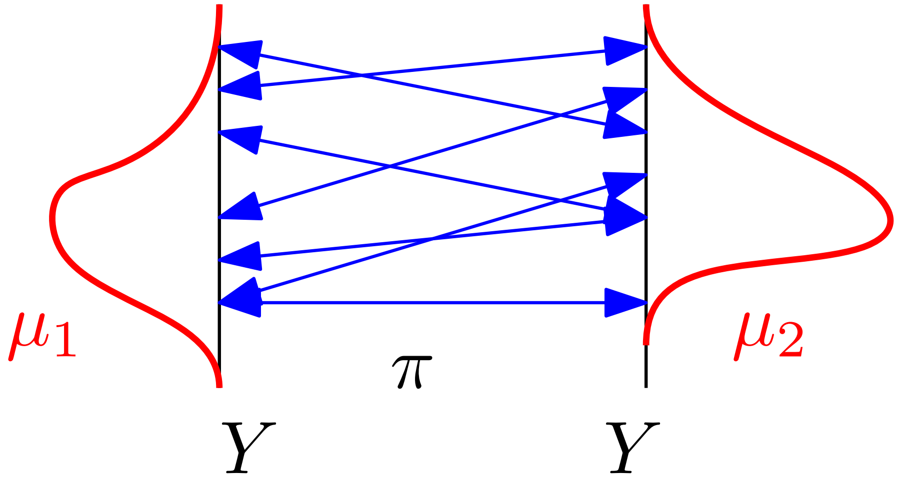

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Probabilities on $Y \times Y$ with marginals μ_1, μ_2.

$\pi(A \times Y) = \mu_1(A); \pi(Y \times B) = \mu_2(B);$

$$\mathcal{T}_c(\mu_1, \mu_2) = \min_{\pi} \left\{ \int_{Y \times Y} c(y_1, y_2)\, \pi(\mathrm{d}y_1, \mathrm{d}y_2) \ : \ \pi \in \Pi(\mu_1, \mu_2) \right\}$$

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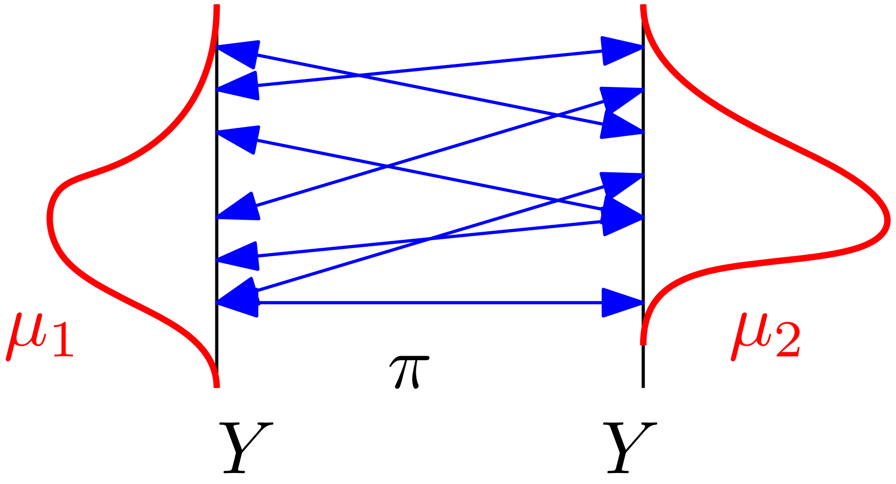

Question: how to extend c into $\mathcal{T}_c : \mathcal{P}(Y) \times \mathcal{P}(Y) \to [0, +\infty]$

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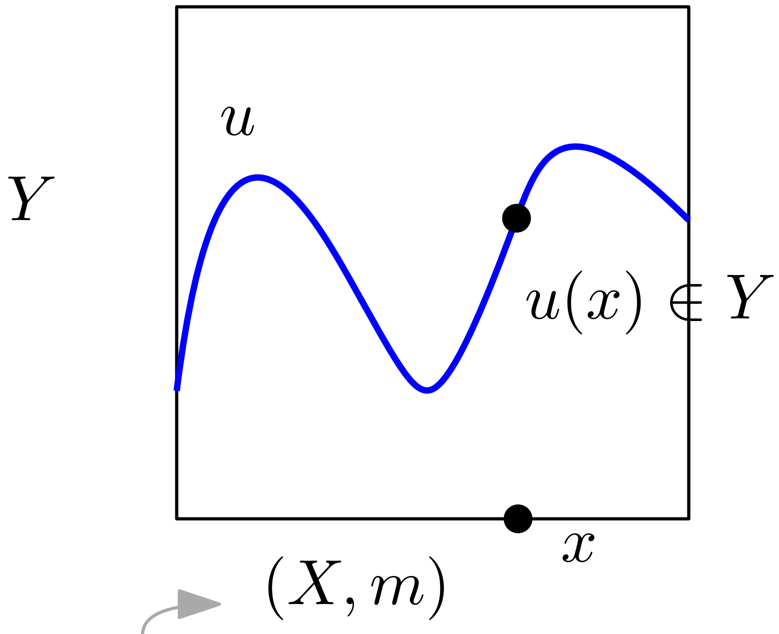
$$\mathcal{T}_c(\mu_1, \mu_2) = \min_{\pi} \left\{ \int_{Y \times Y} c(y_1, y_2)\, \pi(\mathrm{d}y_1, \mathrm{d}y_2) \ : \ \pi \in \Pi(\mu_1, \mu_2) \right\}$$

Theorem. $\mathcal{T}_c$ is the largest **convex** and **lower semi continuous** functional on $\mathcal{P}(Y) \times \mathcal{P}(Y)$ such that $\mathcal{T}_c(\delta_{y_1}, \delta_{y_2}) = c(y_1, y_2)$ for any y_1, y_2.

w.r.t. narrow convergence if c l.s.c. and, e.g. Y polish space

Savaré and Sodini (2022). A simple relaxation approach to duality for Optimal Transport problems in completely regular spaces.

Today's question

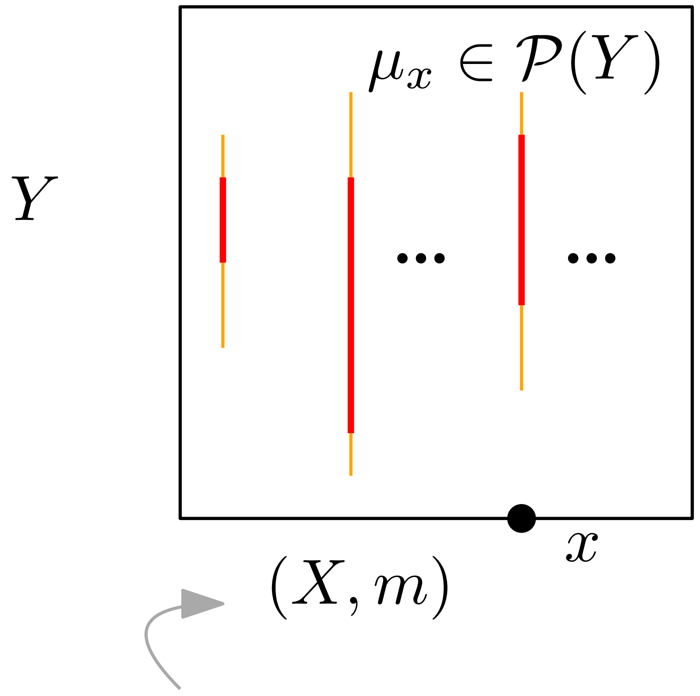


measure on X

Maps $u : X \to Y$, equivalent if equal m-a.e.

$$\boxed{E : L^0(X, Y, m) \to [0, +\infty]}$$

Today's question



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Want to extend to $L^0(X, \mathcal{P}(Y), m)$

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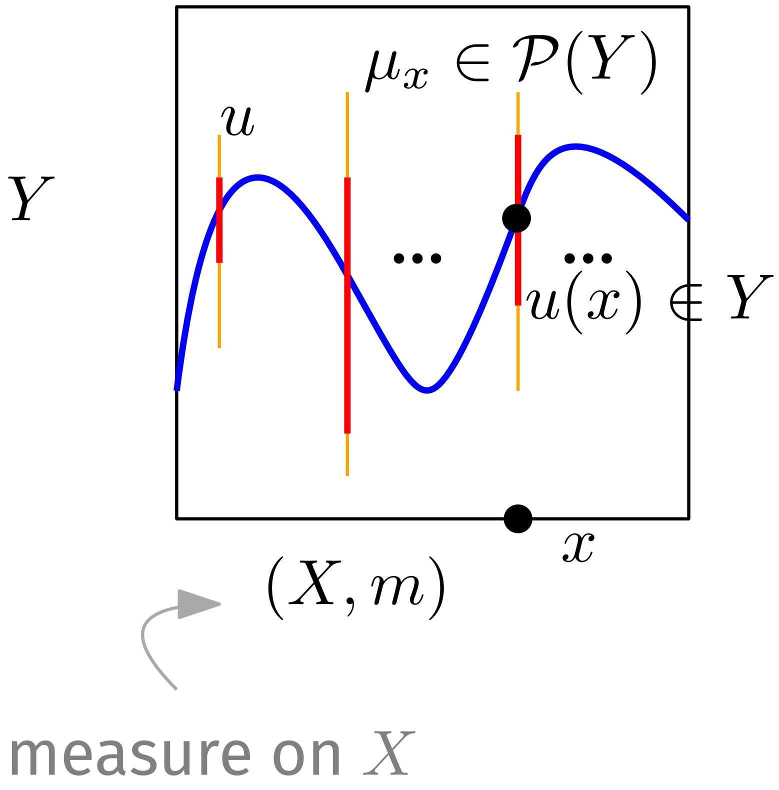

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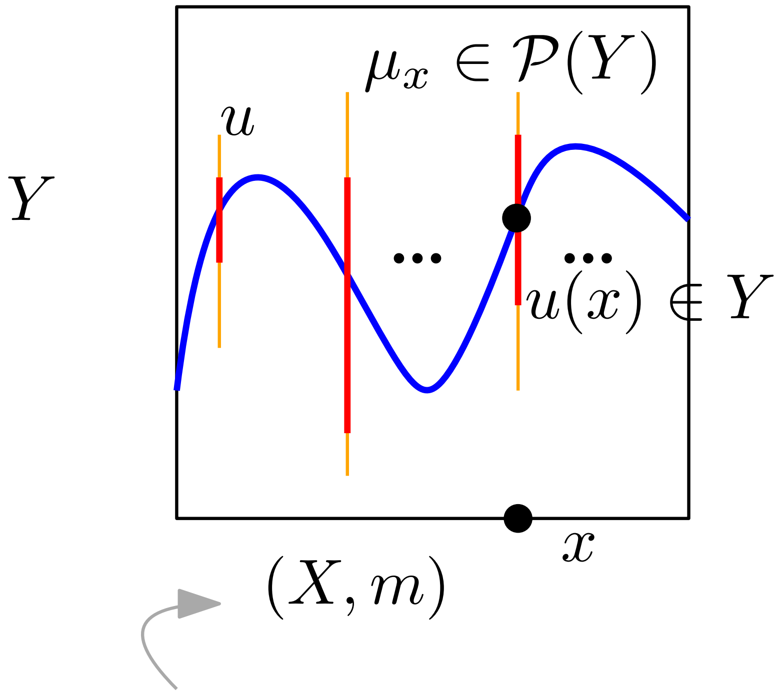
$$\boxed{E : L^0(X, Y, m) \to [0, +\infty]}$$

Want to extend to $L^0(X, \mathcal{P}(Y), m)$

Define $\mu_u : x \mapsto \delta_{u(x)}$.

Today's question



measure on X

Maps $u : X \to Y$, equivalent if equal m-a.e.

$$\boxed{E : L^0(X, Y, m) \to [0, +\infty]}$$

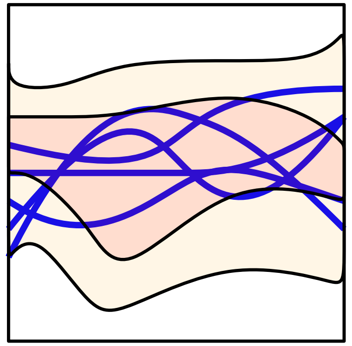
Want to extend to $L^0(X, \mathcal{P}(Y), m)$

Define $\mu_u : x \mapsto \delta_{u(x)}$.

Question. What is the largest **convex** and **lower semi continuous** functional

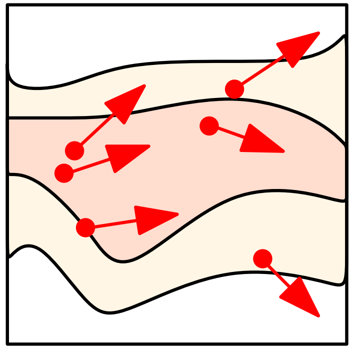
$$\mathcal{T} : L^0(X, \mathcal{P}(Y), m) \to [0, +\infty]$$

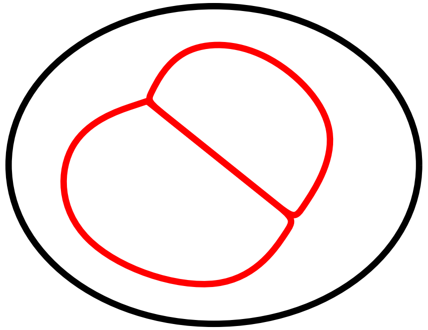
such that $\mathcal{T}(\mu_u) = E(u)$ for all u? (for which topology?)



1 - The Lagrangian lifting or optimal transport with an infinity of marginals

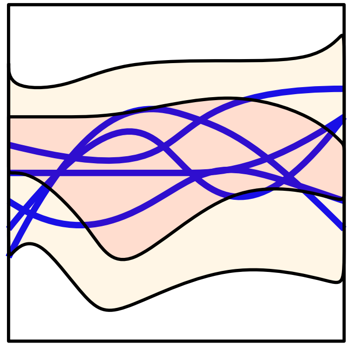
2 - The Eulerian lifting





3 - Understanding the difference: localization of functionals

X, Y polish (metric, complete, separable) spaces.



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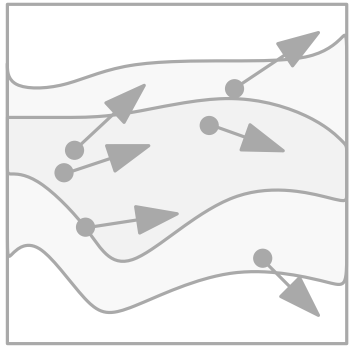
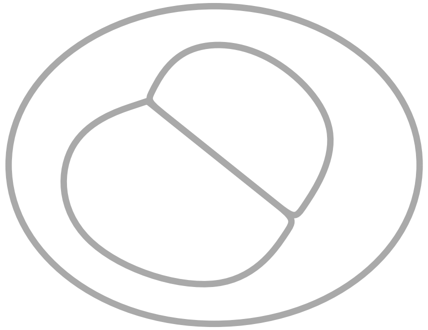
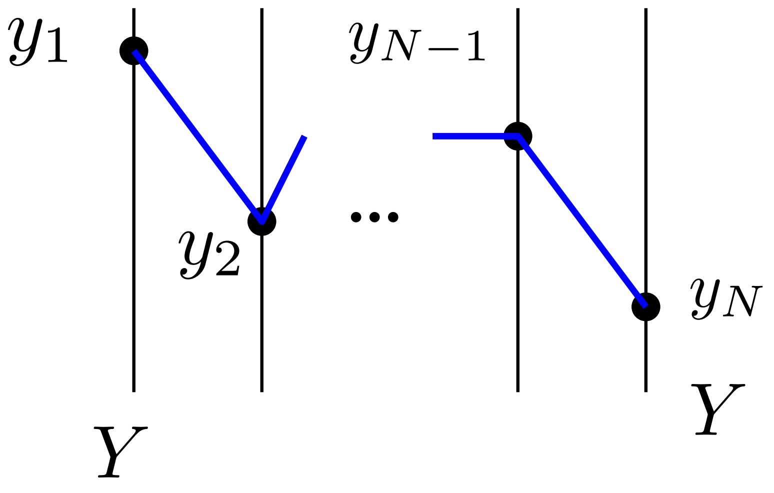




3 - Understanding the difference: localization of functionals

Optimal transport with several marginals

Transport problem with N marginals: $c : Y^N \to [0, +\infty]$



cost $c(y_1, \ldots, y_N)$

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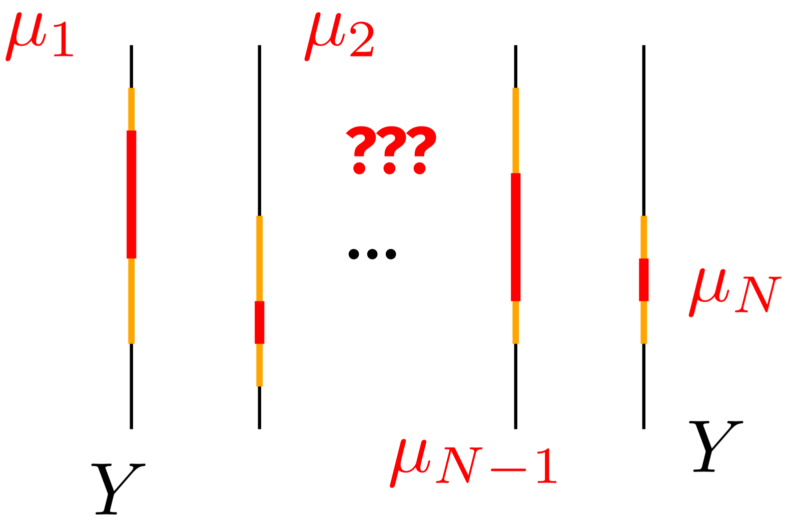

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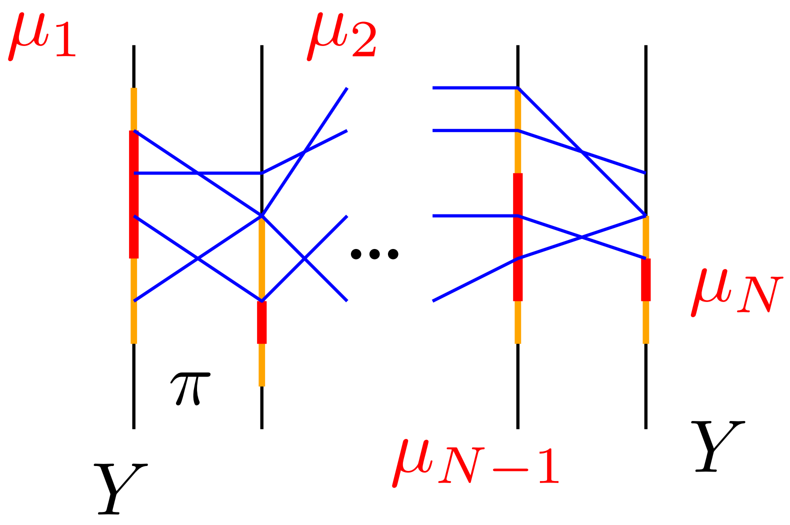

Question: how to extend c into $\mathcal{T}_c : \mathcal{P}(Y)^N \to [0, +\infty]$

Probabilities on Y^N with marginals $\mu_1, \dots, \mu_N$

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Largest convex l.s.c. such that $\mathcal{T}_c(\delta_{y_1}, \dots, \delta_{y_N}) = c(y_1, \dots, y_N)$.

Optimal transport with several marginals

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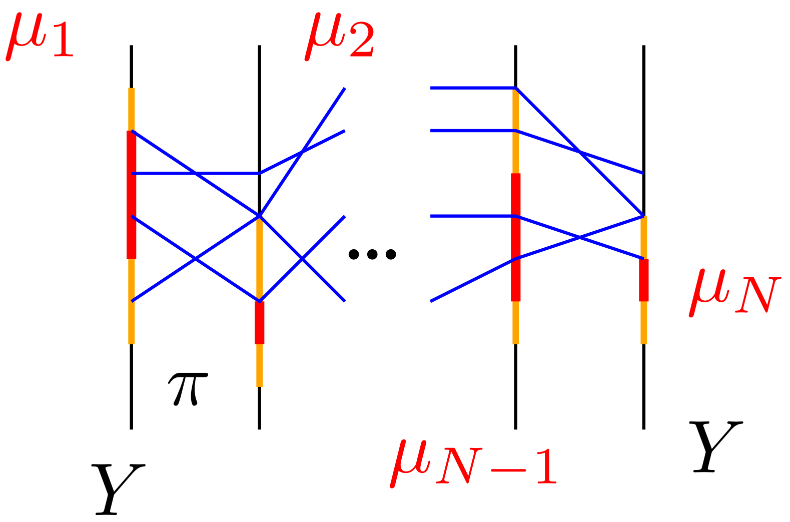

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Idea: take limit $N \to +\infty$: indexing set $\{1, \dots, N\}$ becomes X

Optimal transport with several marginals

Tra marginals: $c : Y^N \to [0, +\infty]$

Y^N becomes $X \to Y$, and c becomes E

$\mu_1, \ldots, \mu_N \in \mathcal{P}(Y)^N$ becomes $\mu : X \to \mathcal{P}(Y)$

π becomes Q probability on maps $X \to Y$.

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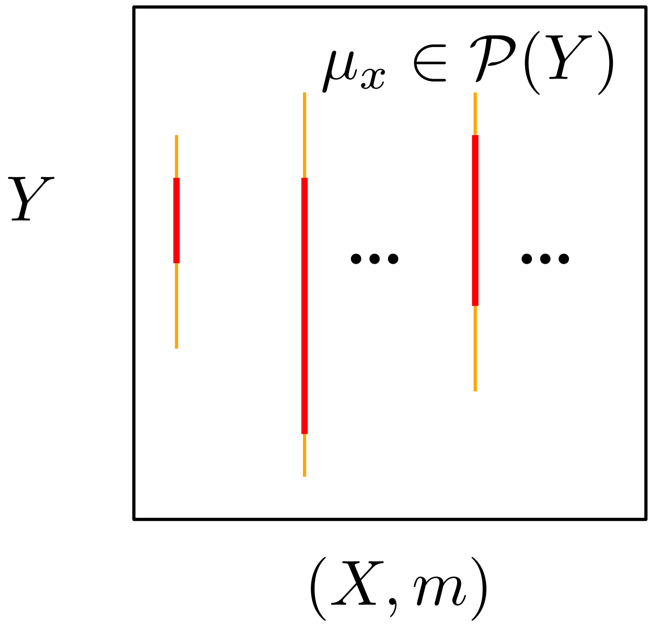
Maps of measures and measures on maps



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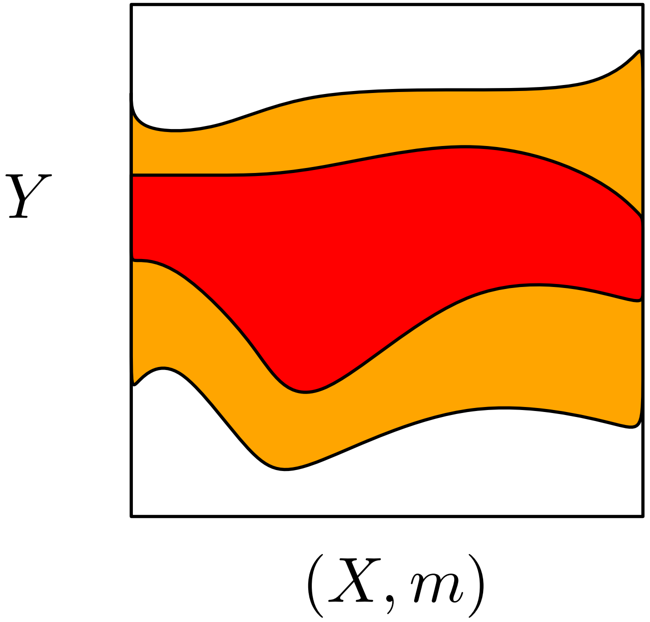
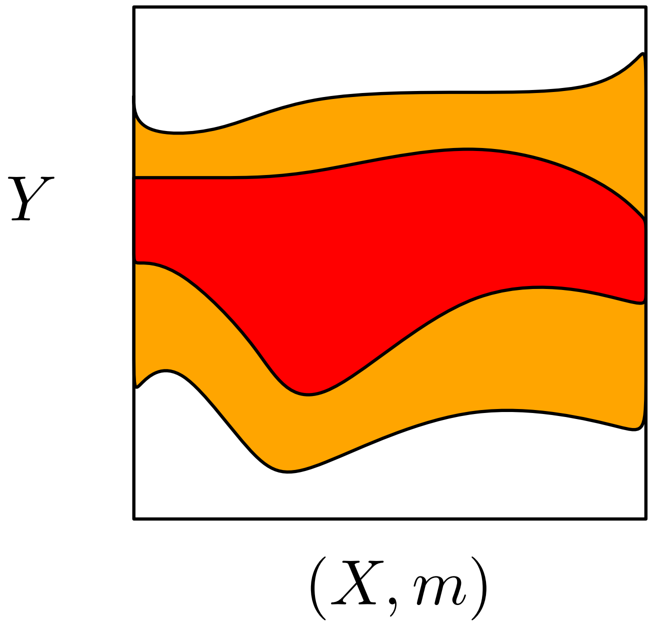


View μ as measure on $X \times Y$ by

$$\int_{X\times Y} \varphi \,\mathrm{d}\mu = \int_X \left(\int_Y \varphi(x,y)\,\mathrm{d}\mu_x(y) \right) \mathrm{d}m(x)$$

Maps of measures and measures on maps



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Theorem. (disintegration). As sets, $L^0(X, \mathcal{P}(Y), m)$ coincides with measures on $X \times Y$ whose first marginal is m.

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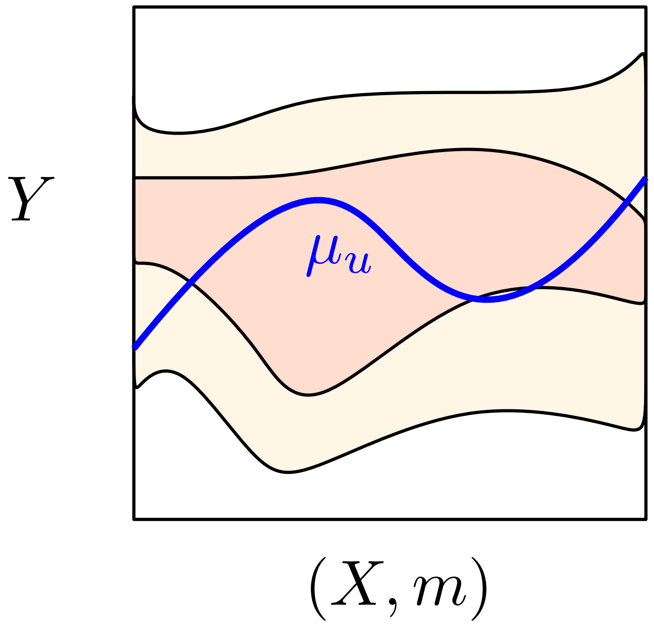

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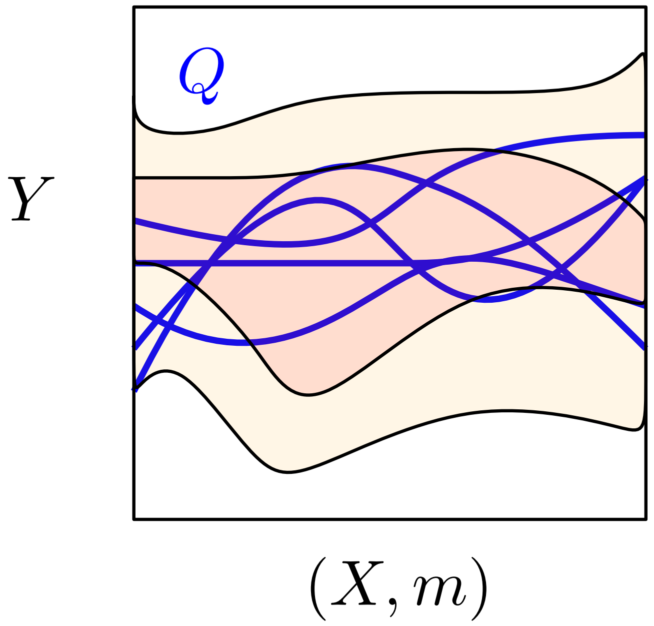

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Recall: $\mu_u : x \mapsto \delta_{u(x)}$.

Definition. $Q \in \mathcal{P}(L^0(X, Y, m))$ belongs to $\Pi(\mu)$ if

$$\mu = \int_{L^0(X,Y,m)} \mu_u \, \mathrm{d}Q(u).$$

(Intuition: if $u \sim Q$ then $u(x) \sim \mu_x$ for m-a.e. x.)

Proposition. $\Pi(\mu)$ if never empty (if X, Y polish spaces).

Multimarginal optimal transport with an infinity of marginals

- $E : L^0(X, Y, m) \to [0, +\infty]$ lower semi continuous,
- μ measure on $X \times Y$ with first marginal m.

Definition.

$$\mathcal{T}_E(\mu) = \inf_Q \left\{ \int_{L^0(X,Y,m)} E(u) \, \mathrm{d}Q(u) \ : \ Q \in \Pi(\mu) \right\}.$$

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Theorem.

Narrow convergence on $\mathcal{M}_+(X \times Y)$

- $\mathcal{T}_E$ is always convex.
- Under additional assumption, it is the largest convex and l.s.c. functional such that

$$\forall u, \quad \mathcal{T}_E(\mu_u) = E(u)$$

see next slides

Idea of the proof

Any $\mathcal{T}$ convex l.s.c. such that $\mathcal{T}(\mu_u) = E(u)$

Idea of the proof

Any $\mathcal{T}$ convex l.s.c. such that $\mathcal{T}(\mu_u) = E(u)$

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Using def of $Q \in \Pi(\mu)$

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$$\leq \int_{L^0} \mathcal{T}(\mu_u) \,\mathrm{d}Q(u)$$

Jensen

Idea of the proof

Any $\mathcal{T}$ convex l.s.c. such that $\mathcal{T}(\mu_u) = E(u)$

$$\begin{aligned} \mathcal{T}(\mu) &= \mathcal{T}\left(\int_{L^0} \mu_u \, \mathrm{d}Q(u)\right) && \text{Using def of } Q \in \Pi(\mu) \\ &\leq \int_{L^0} \mathcal{T}(\mu_u) \, \mathrm{d}Q(u) && \text{Jensen} \\ &= \int_{L^0} E(u) \mathrm{d}Q(u) \end{aligned}$$

Idea of the proof

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\end{aligned}
$$

Left to do: prove that $\mathcal{T}_E$ is lower semi continuous.

Assumption on E

To guarantee existence of optimal $Q \in \Pi(\mu)$ and l.s.c. of $\mathcal{T}_E$.

Assumption.
E is l.s.c. and

the functional $u \mapsto E(u)$

has compact sublevel sets in $L^0(X, Y, m)$.

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Think $E(u) = \int |\nabla u|^2$ ✓

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"Coercivity of E + Tightness of μ"

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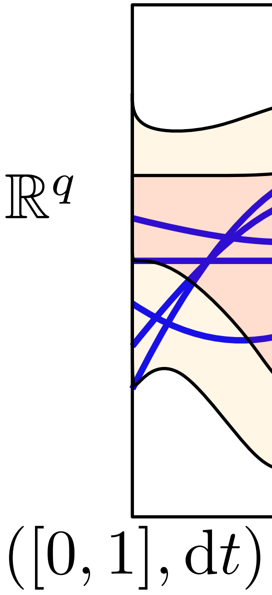
“Coercivity of E + Tightness of μ”

Think $E(u) = \int |\nabla u|^2$ ✓

Remark. If X finite, no assumption needed on E besides l.s.c.

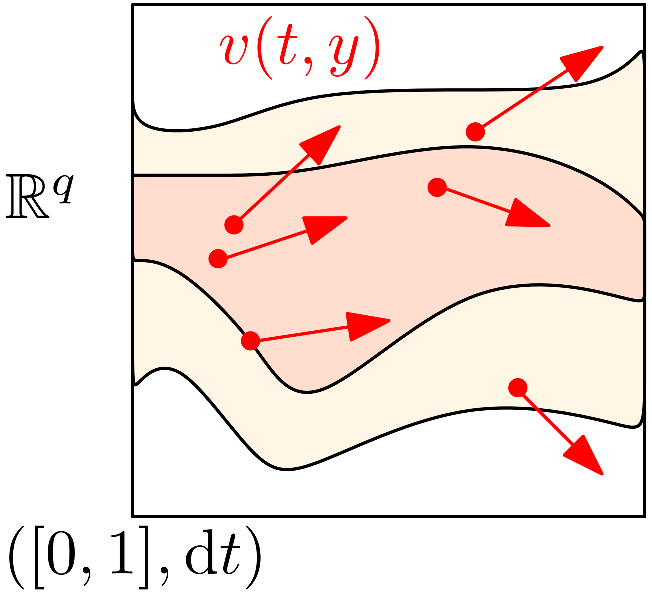
Curves of measures and measures on curves



$$E(u) = \int_0^1 |\dot{u}_t|^p \, \mathrm{d}t$$

The previous result holds and there exists an optimal Q.

Curves of measures and measures on curves



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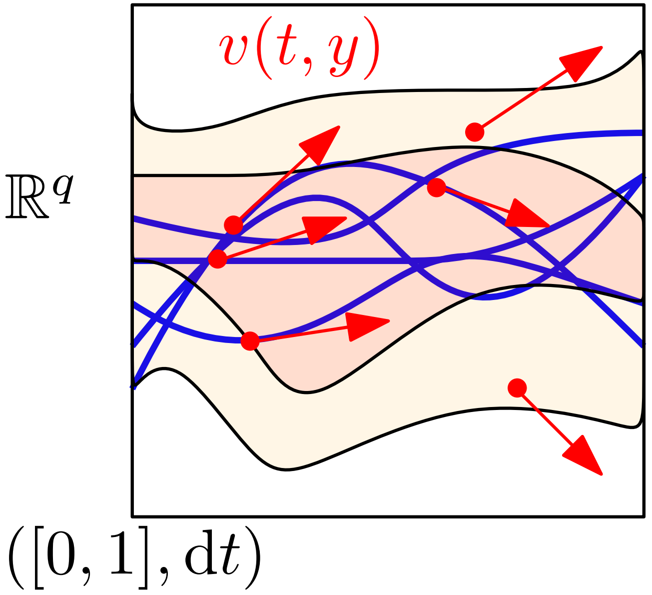
The previous result holds and there exists an optimal Q.

Theorem. $$\mathcal{T}_E(\mu) = \min_v \left\{ \int_0^1 \int_{\mathbb{R}^q} |v(t,y)|^p \, \mathrm{d}\mu_t(y)\mathrm{d}t \ : \ \partial_t \mu + \mathrm{div}_y(v\mu) = 0 \right\}.$$

Lisini (2007). Characterization of absolutely continuous curves in Wasserstein spaces.
Ambrosio, Gigli and Savaré (2008). Gradient flows in metric spaces and in the space of probability measures.

Curves of measures and measures on curves



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At optimality: $\dot{u}_t = v(t,u_t)$ for Q-a.e. u.

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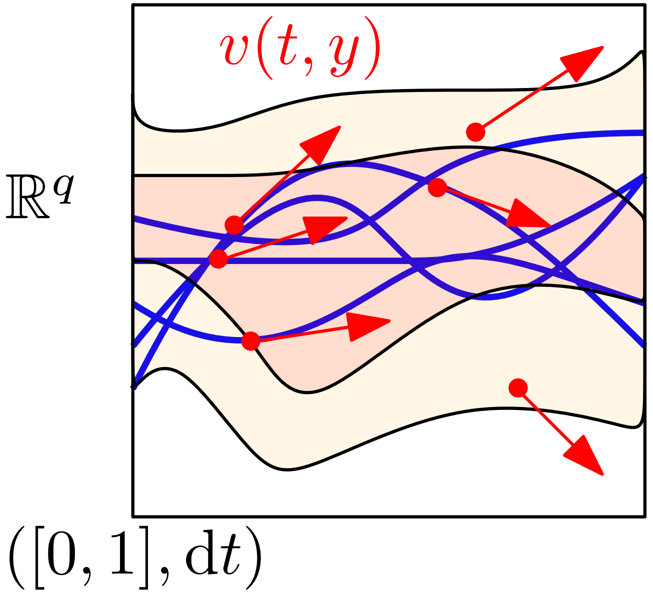

$$E(u) = \int_0^1 |\dot{u}_t|^p \, \mathrm{d}t$$

$E(u) = \int_0^1 |\ddot{u}|^2$ for **splines.**

The previous result holds and there exists an optimal Q.

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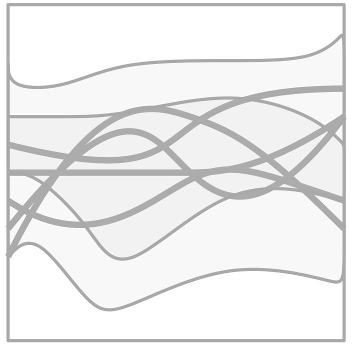
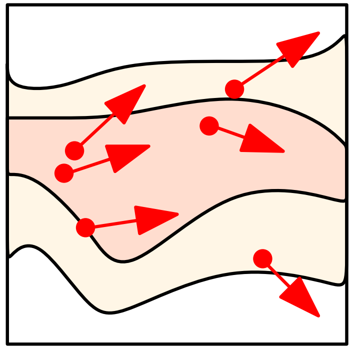
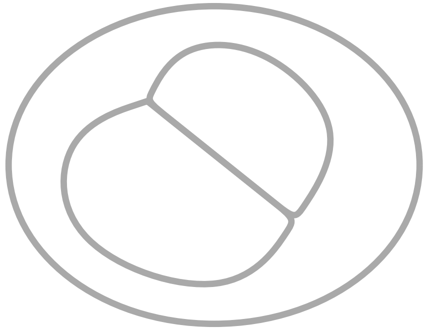
Ambrosio, Gigli and Savaré (2008). Gradient flows in metric spaces and in the space of probability measures.
Benamou, Gallouët, Vialard (2019). Second-order models for optimal transport and cubic splines on the Wasserstein space.
Chen, Conforti, Georgiou (2018). Measure-valued spline curves: An optimal transport viewpoint.



1 - The Lagrangian lifting or optimal transport with an infinity of marginals

2 - The Eulerian lifting

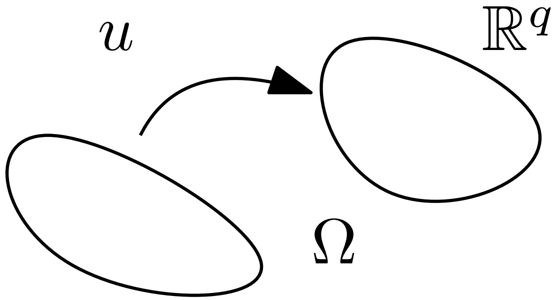




3 - Understanding the difference: localization of functionals

The Eulerian lifting

$X = \Omega \subset \mathbb{R}^d$ with Lebesgue measure, $Y = \mathbb{R}^q$, and W **convex**



$$E(u) = \int_\Omega W(\nabla u(x))\mathrm{d}x$$

Vogt, Haase, Bednarski and Lellmann (2020). On the connection between dynamical optimal transport and functional lifting.

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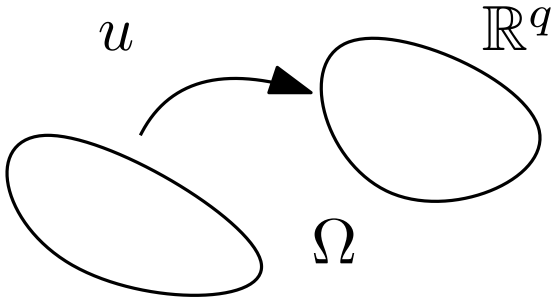

$$E(u) = \int_\Omega W(\nabla u(x)) \mathrm{d}x$$

Definition. We define $\mathcal{T}_{E,\mathrm{Eul}}(\mu)$ as

$$\min_v \left\{ \int_\Omega \int_{\mathbb{R}^q} W(v(x,y))\, \mathrm{d}\mu_x(y) \mathrm{d}x \quad \text{s.t. } \nabla_x \mu + \mathrm{div}_y(v\mu) = 0 \right\}$$

$v : \Omega \times \mathbb{R}^q \to \mathbb{R}^{q \times d}$ “density of Jacobian matrix”.

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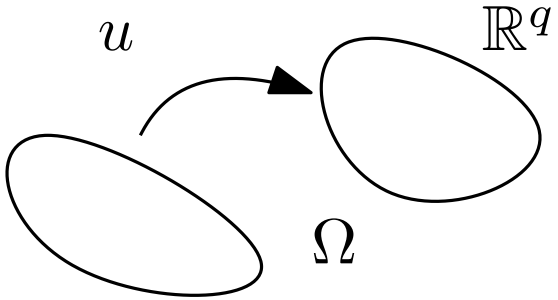

$$E(u) = \int_\Omega W(\nabla u(x))\mathrm{d}x + \int_\Omega f(x, u(x))\,\mathrm{d}x$$

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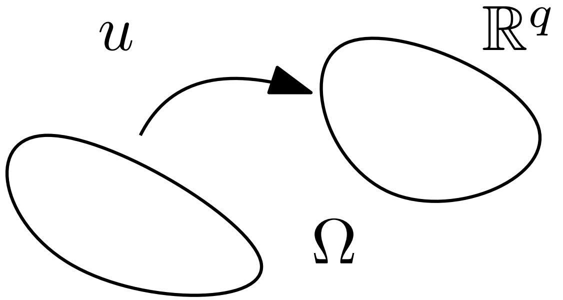

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$v : \Omega \times \mathbb{R}^q \to \mathbb{R}^{q\times d}$ “density of Jacobian matrix”.

Remark. To have a convex formulation: $(\mu, v) \leftrightarrow (\mu, v\mu)$.

Vogt, Haase, Bednarski and Lellmann (2020). On the connection between dynamical optimal transport and functional lifting.

Example: harmonic maps valued in the Wasserstein space

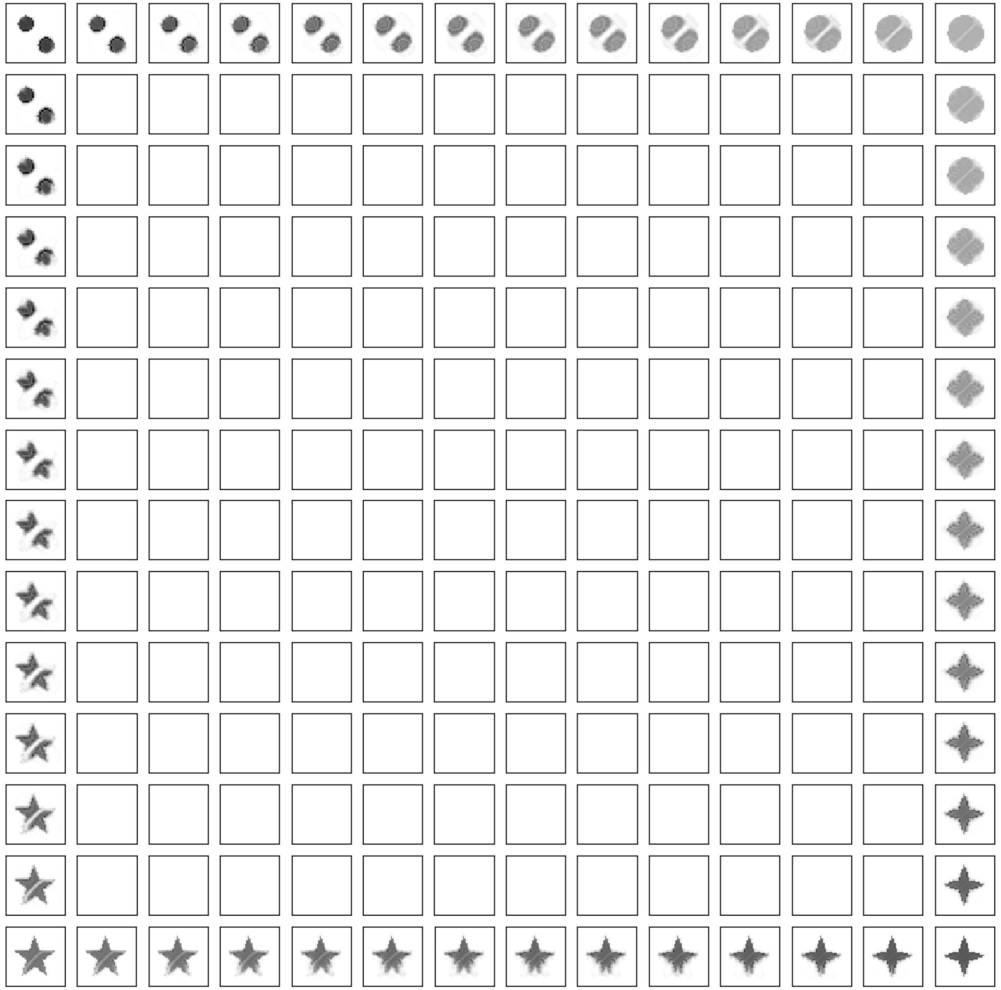
$$E(u) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x$$

Dirichlet problem.

$$\min_\mu \{ \mathcal{T}_{E,\mathrm{Eul}}(\mu)$$

$$\mu_x \text{ given for } x \in \partial\Omega \}$$

Solutions are **harmonic** maps.



Lavenant (2019). Harmonic mappings valued in the Wasserstein space.

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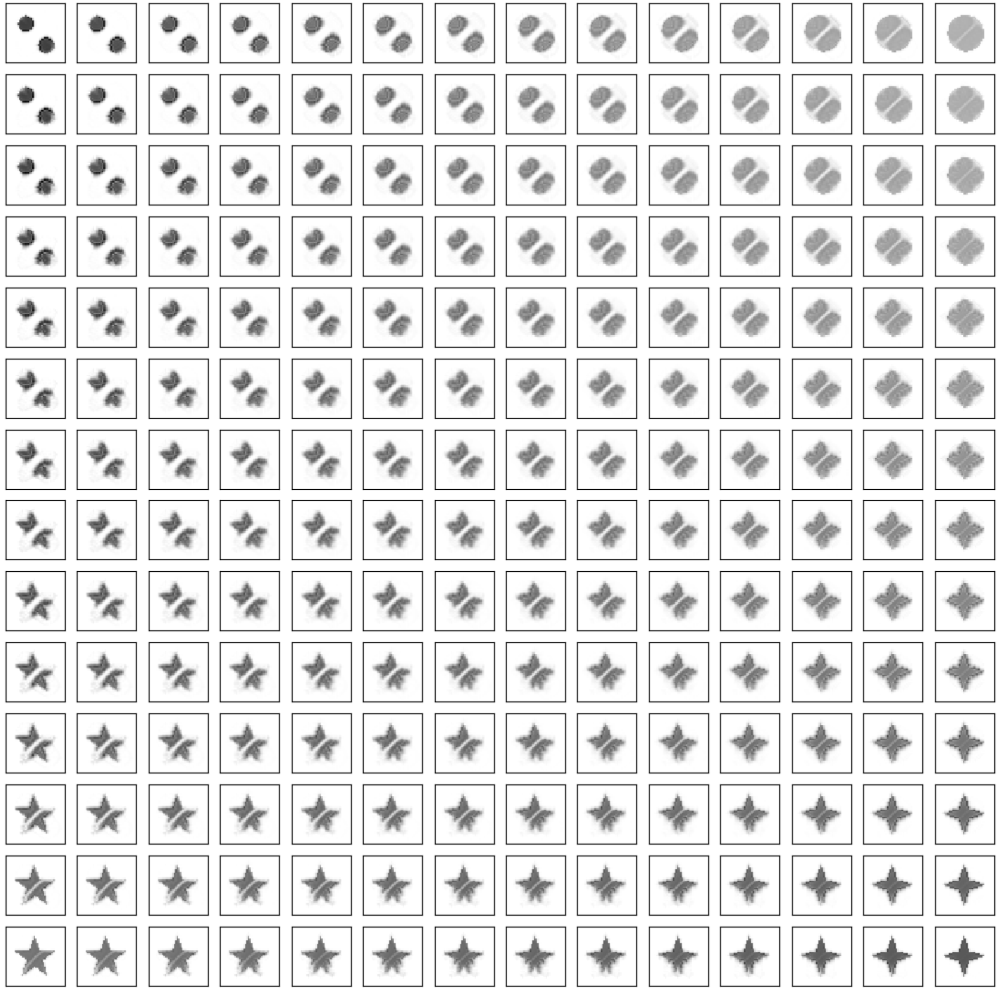

Solutions are **harmonic** maps.

Theorem. If $x \in \partial\Omega \to \mu_x$ is Lipschitz for $(\mathcal{P}(Y), W_2)$ then there exists a minimizer.

Lavenant (2019). Harmonic mappings valued in the Wasserstein space.

Some properties

Restrict to the case $E(u) = \int_\Omega W(\nabla u)$.

Proposition. The functional $\mu \to \mathcal{T}_{E,\mathrm{Eul}}(\mu)$ is convex and l.s.c.

under **assumption** that W grows at least like $|v|^p$ for some $p \geq 1$.

Proposition. $\mathcal{T}_{E,\mathrm{Eul}}(\mu_u) = E(u)$ for any u.

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Consequence: $\mathcal{T}_{E,\mathrm{Eul}} \leq \mathcal{T}_E$. ⟶ Equal if $\Omega = [0,1]$ is a segment!

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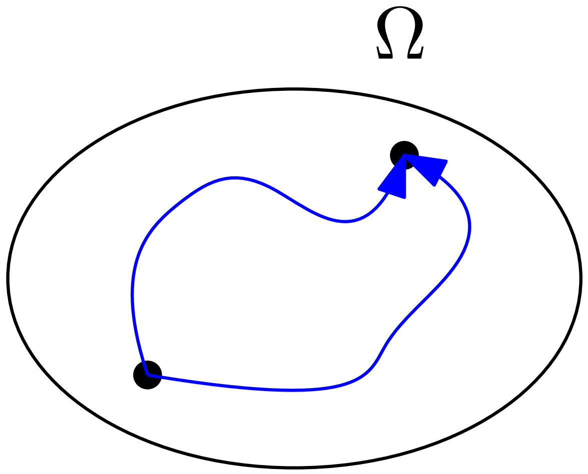
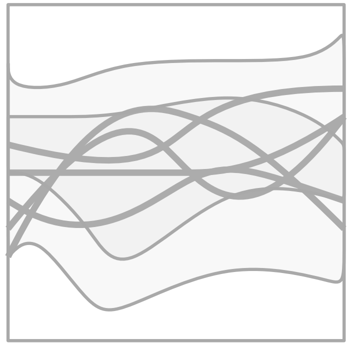


Consequence: $\mathcal{T}_{E,\mathrm{Eul}} \leq \mathcal{T}_E$.

If equal then $\nabla u(x) = v(x, u(x))$ for Q-a.e. u.

✗ **Should be different in general**



1 - The Lagrangian lifting or optimal transport with an infinity of marginals

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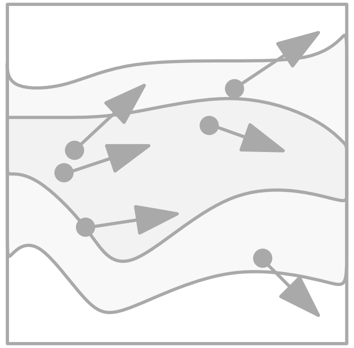
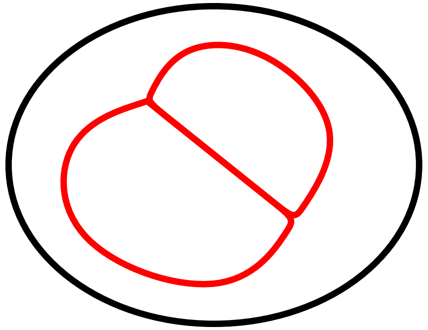
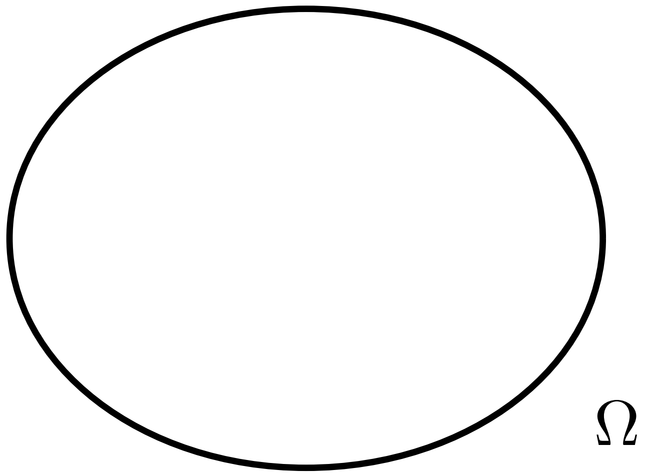




3 - Understanding the difference: localization of functionals

Localization of functionals

Previously E depends on u: $E(u) = \int_\Omega W(\nabla u).$



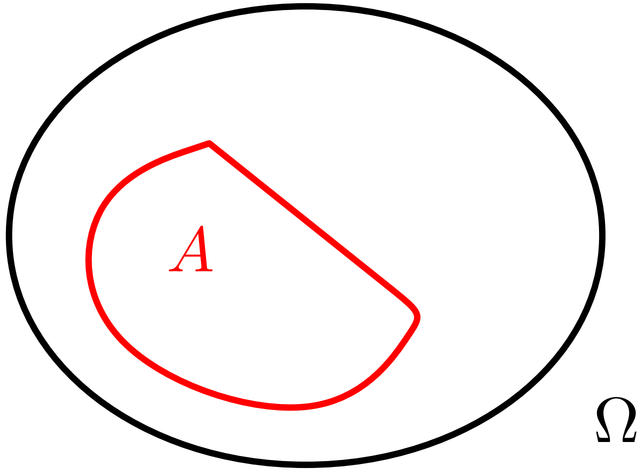
Localization of functionals

Previously E depends on u: $$E(u) = \int_\Omega W(\nabla u).$$

Localized version: $$E(u, A) = \int_A W(\nabla u)$$

map

Open set $A \subseteq \Omega$



Dal Maso (2012). An introduction to Γ-convergence.

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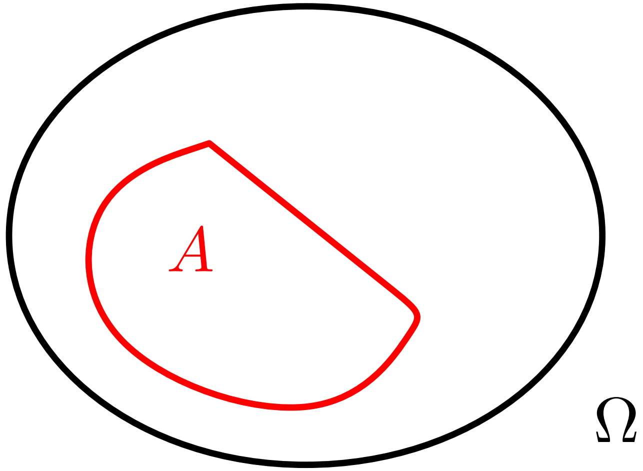


Definition. A localized functional E is

- **convex and l.s.c.** if $E(\cdot, A)$ is convex and l.s.c for any A.

Dal Maso (2012). An introduction to Γ-convergence.

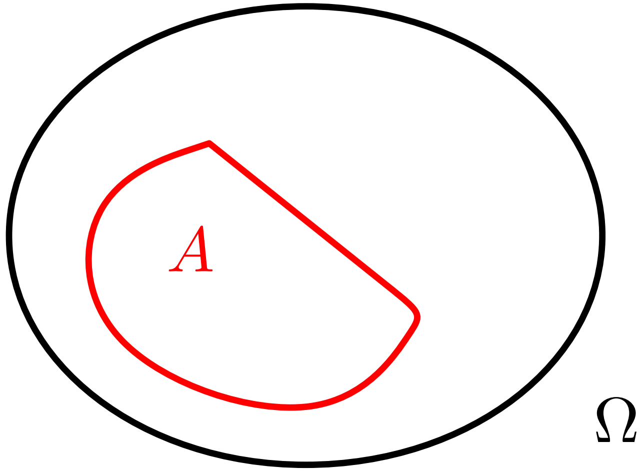
Localization of functionals

Previously E depends on u: $E(u) = \int_\Omega W(\nabla u).$

Localized version: $E(u, {\color{red}A}) = \int_{\color{red}A} W(\nabla u)$

map

Open set $A \subseteq \Omega$



Definition. A localized functional E is

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- **local** if $E(u, A)$ only depends on the restriction of u to A.

Dal Maso (2012). An introduction to Γ-convergence.

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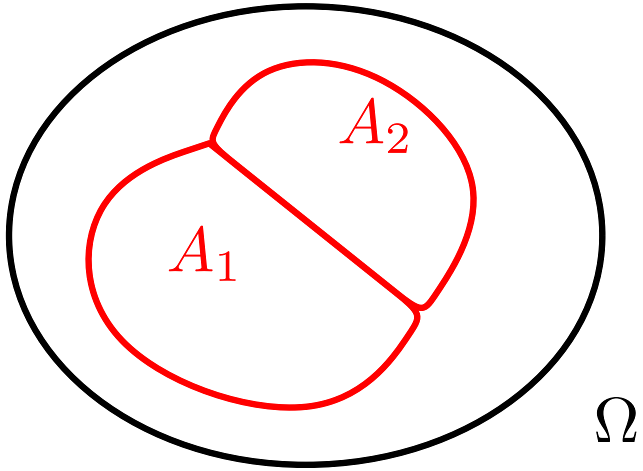

Definition. A localized functional E is

- **convex and l.s.c.** if $E(\cdot, A)$ is convex and l.s.c for any A.
- **local** if $E(u, A)$ only depends on the restriction of u to A.
- **additive** if for any u, A_1, A_2 with A_1, A_2 disjoint:

$$E(u, A_1 \cup A_2) = E(u, A_1) + E(u, A_2).$$

Possible to extend to **countably** additive

Dal Maso (2012). An introduction to Γ-convergence.

For the Eulerian lifting

Under **assumption** that W grows at least like $|v|^p$ for some $p \geq 1$.

The functional $$E(u, A) = \int_A W(\nabla u)$$

is **convex**, **l.s.c.**, **local** and a **countably additive**.

For the Eulerian lifting

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Localized Eulerian lifting:

$$\mathcal{T}_{E,\mathrm{Eul}}(\mu, A) = \min_v \left\{ \int_A \int_{\mathbb{R}^q} W(v(x,y))\,\mathrm{d}\mu_x(y)\mathrm{d}x \ \text{ s.t. } \ \nabla_x \mu + \mathrm{div}_y(v\mu) = 0 \right\}$$

for $v : A \times \mathbb{R}^q \to \mathbb{R}^{q \times d}$.

For the Eulerian lifting

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for $v : A \times \mathbb{R}^q \to \mathbb{R}^{q \times d}$.

Proposition. This lifting is **convex**, **l.s.c.**, **local** and a **countably additive**.

For the Lagrangian lifting

Localized version:

$$\mathcal{T}_E(\mu, A) = \inf_Q \left\{ \int_{L^0(X,Y,m)} E(u, A)\, \mathrm{d}Q(u) \ : \ Q \in \Pi(\mu) \right\}.$$

$\mathcal{T}_E$ is **local** if E is local.

For the Lagrangian lifting

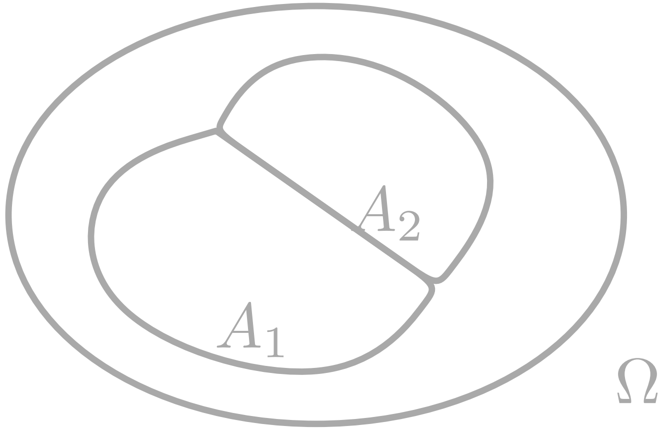
Localized version:

$$\mathcal{T}_E(\mu, A) = \inf_{Q} \left\{ \int_{L^0(X,Y,m)} E(u, A) \, \mathrm{d}Q(u) \ : \ Q \in \Pi(\mu) \right\} .$$

$\mathcal{T}_E$ is **local** if E is local.

Proposition. If E is additive, then $\mathcal{T}_E$ is **superadditive**.

(But **not** additive)



$$\mathcal{T}_E(\mu, A_1 \cup A_2) \geq \mathcal{T}_E(\mu, A_1) + \mathcal{T}_E(\mu, A_2)$$

For the Lagrangian lifting

Localized version:

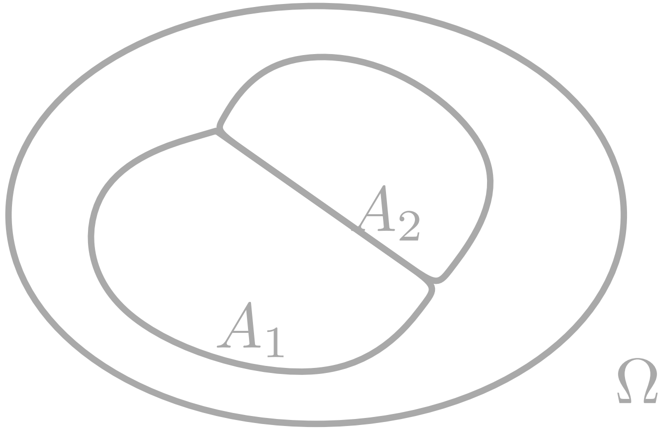
$$\mathcal{T}_E(\mu, A) = \inf_Q \left\{ \int_{L^0(X,Y,m)} E(u, A)\,\mathrm{d}Q(u) \ : \ Q \in \Pi(\mu) \right\}.$$

$\mathcal{T}_E$ is **local** if E is local.

superadditive is sufficient

Proposition. If E is additive, then $\mathcal{T}_E$ is **superadditive**.

(But **not** additive)



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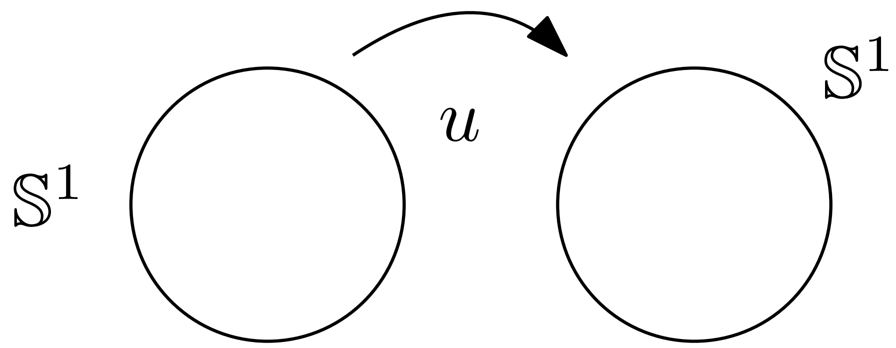
Counterexample for the Lagrangian lifting

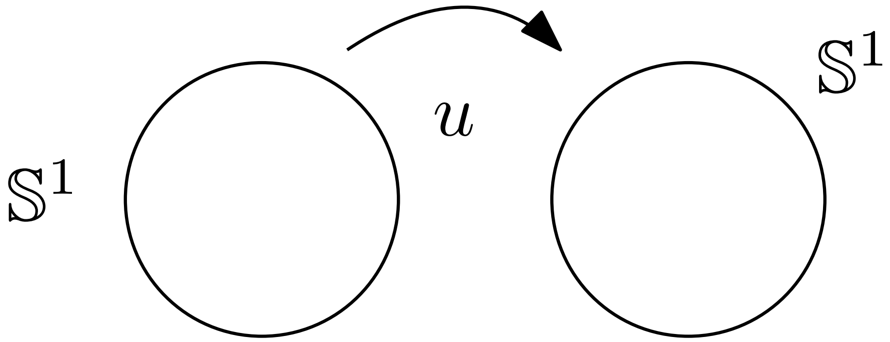


$$E(u, A) = \frac{1}{2} \int_A |\dot{u}_t|^2 \, \mathrm{d}t$$

If $E(u, A) < +\infty$ then u continuous over A.

De Lellis and Spadaro (2011). Q-valued functions revisited.

Counterexample for the Lagrangian lifting



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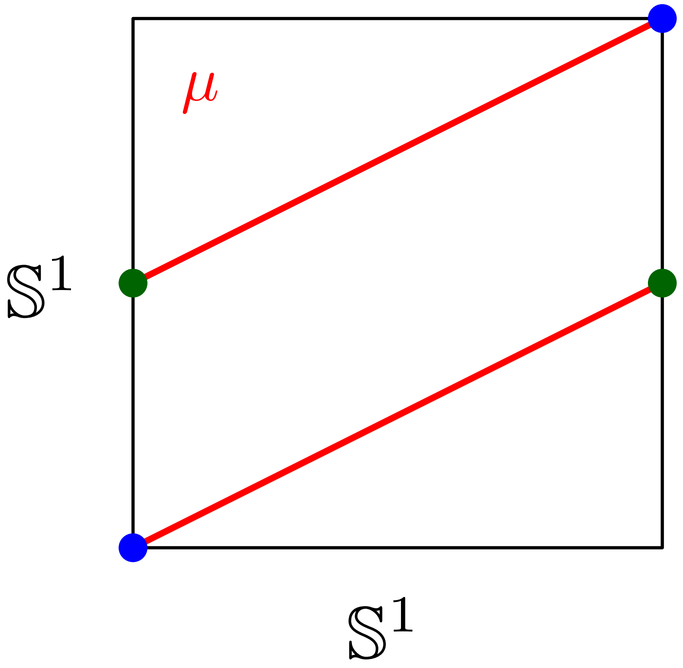
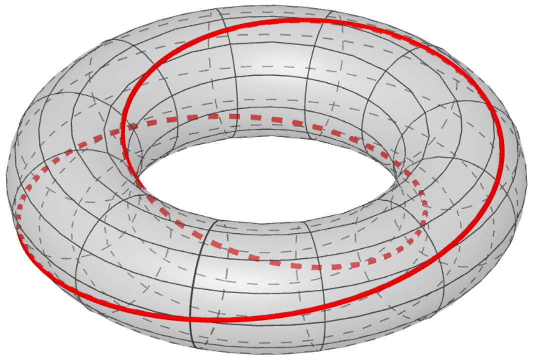
Define

Complex square root

$$\mu_x \to \frac{\delta_{\sqrt{x}} + \delta_{-\sqrt{x}}}{2}$$



$3d$ visualization



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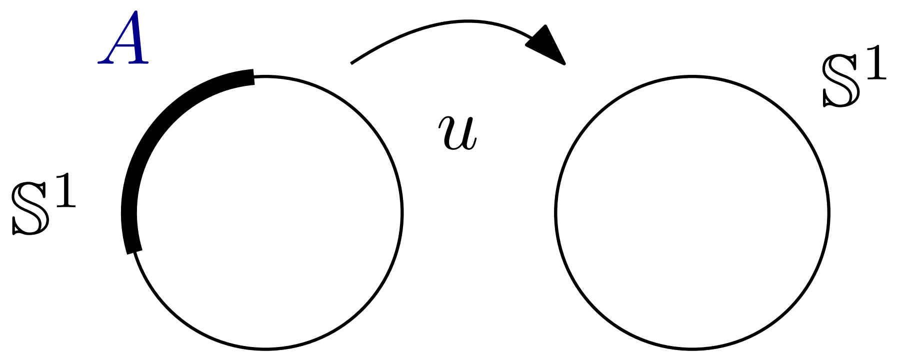
Counterexample for the Lagrangian lifting



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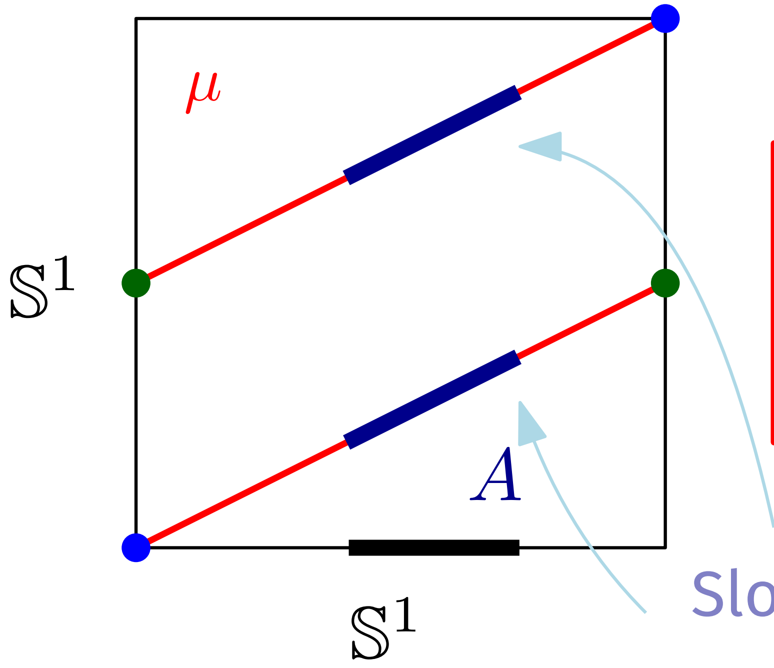

Slope $1/2$

If A not dense in $\mathbb{S}^1$

$$\mathcal{T}_E(\mu, A) \leq \frac{1}{8}|A|$$

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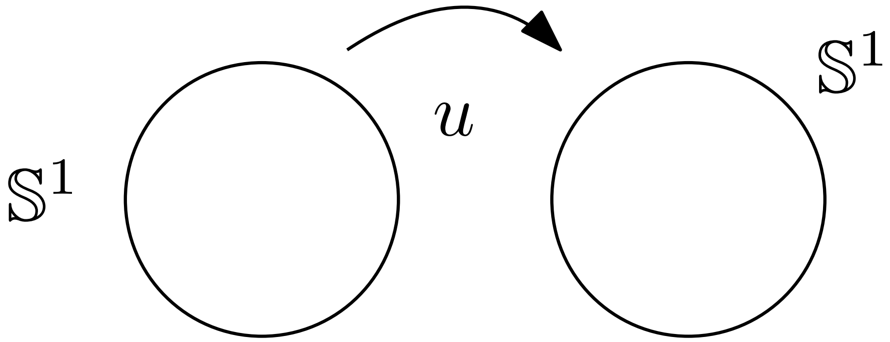
Counterexample for the Lagrangian lifting



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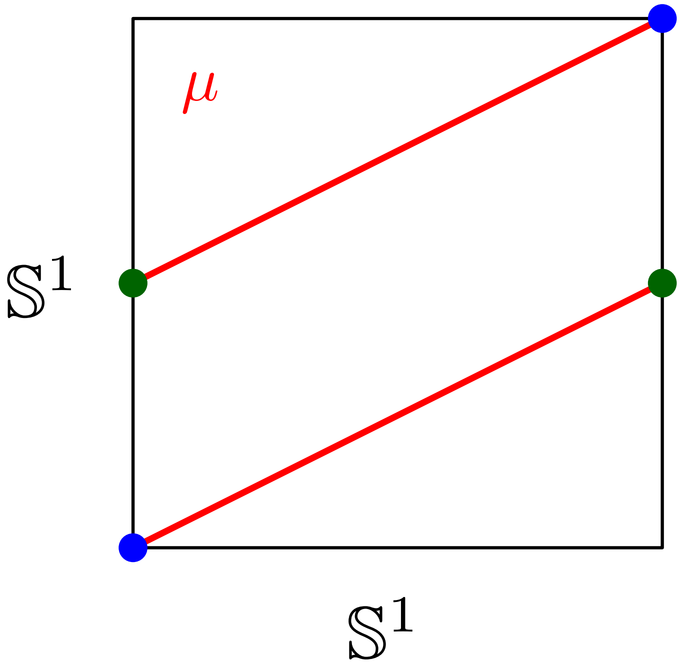

If A not dense in $\mathbb{S}^1$

$$\mathcal{T}_E(\mu, A) \leq \frac{1}{8}|A|$$

$$\mathcal{T}_E(\mu, \mathbb{S}^1) = +\infty.$$

No **continuous** selection of the complex square root exists.

Thus $\mathcal{T}_E(\mu, \cdot)$ is **not** additive.

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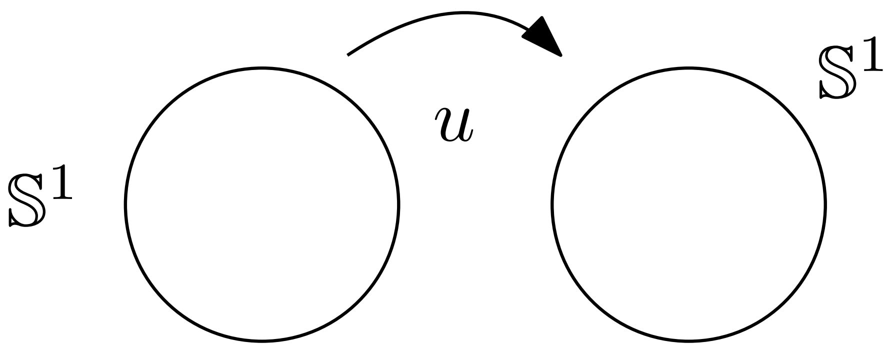
Counterexample for the Lagrangian lifting



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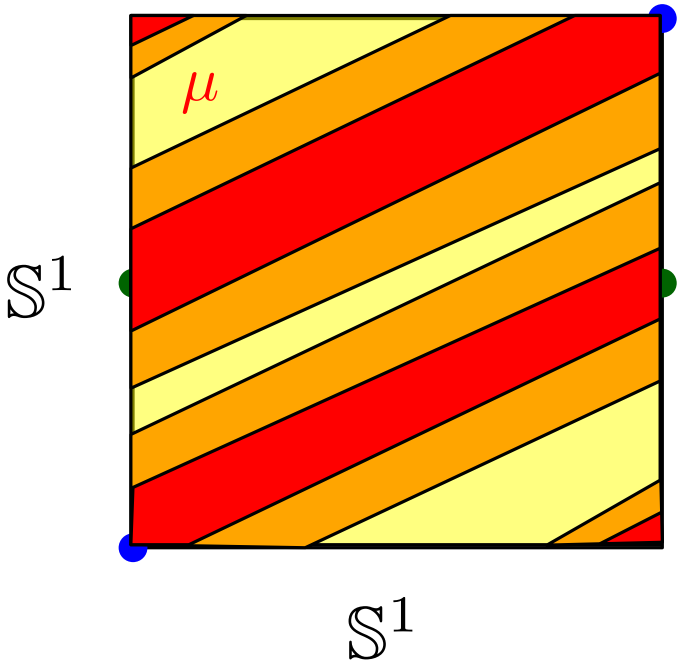

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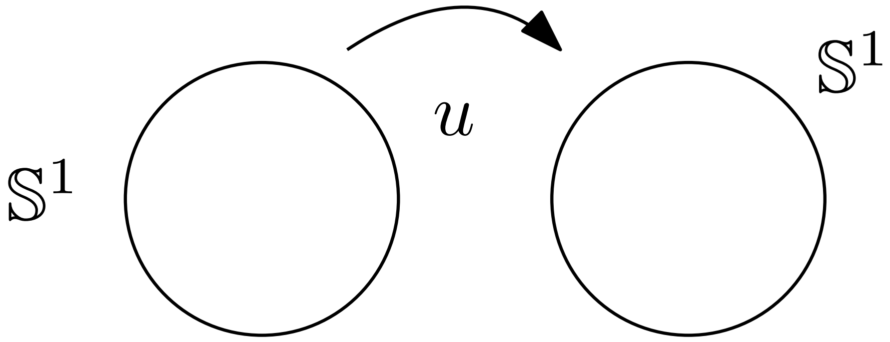
Counterexample for the Lagrangian lifting



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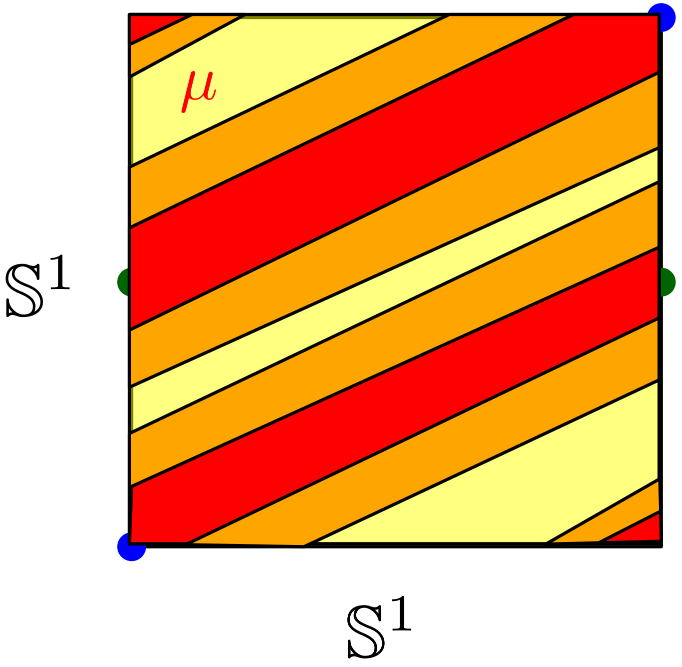

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- Extension to smoothed version.
- Extension to maps $\mathbb{R}^2 \to \mathbb{R}^2$.

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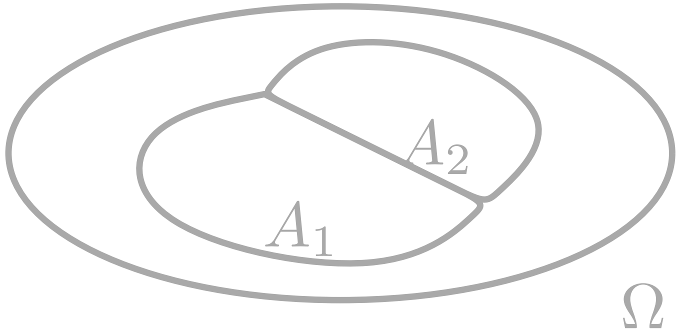
Optimality of the Eulerian lifting

Theorem. For $W : \mathbb{R}^{qd} \to [0, +\infty]$ convex, approximatively radial, define

$$E(u, A) = \int_A W(\nabla u)$$

Then the Eulerian lifting $\mathcal{T}_{E,\mathrm{Eul}}$ is the largest $\mathcal{T}$ **convex**, **l.s.c.**, **subadditive**, increasing and inner regular such that $\mathcal{T}(\mu_u, A) = E(u, A)$.



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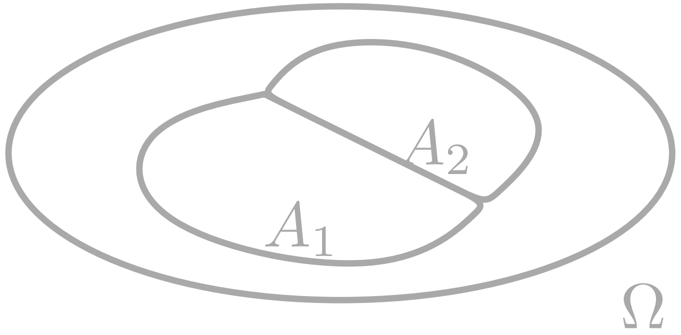

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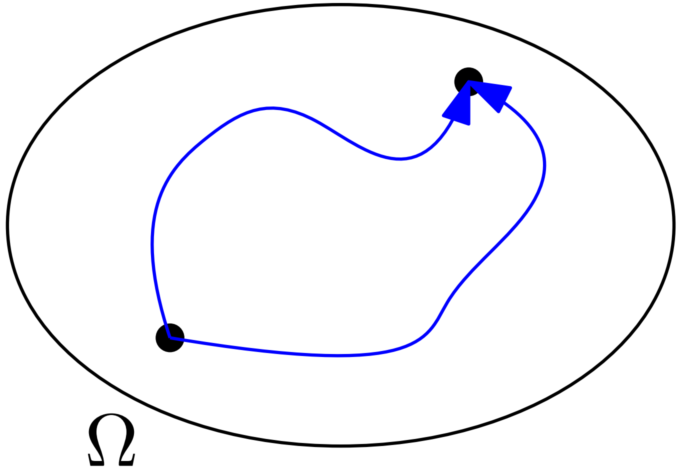
Idea of the proof



If $\mathcal{T}_E(\mu) = \mathcal{T}_{E,\mathrm{Eul}}(\mu)$ then for Q-a.e. map $u : \Omega \to \mathbb{R}^q$

Q optimal for $\mathcal{T}_E$

$$\boxed{\nabla u(x) = v(x, u(x))}$$

v optimal in $\mathcal{T}_{E,\mathrm{Eul}}$

✗ Incompatibility in general

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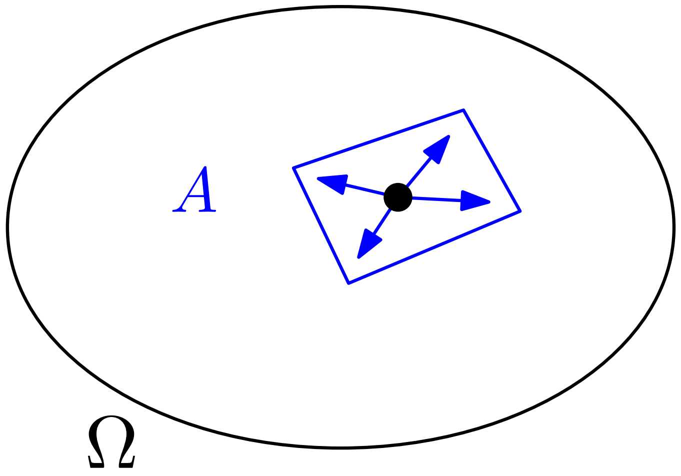

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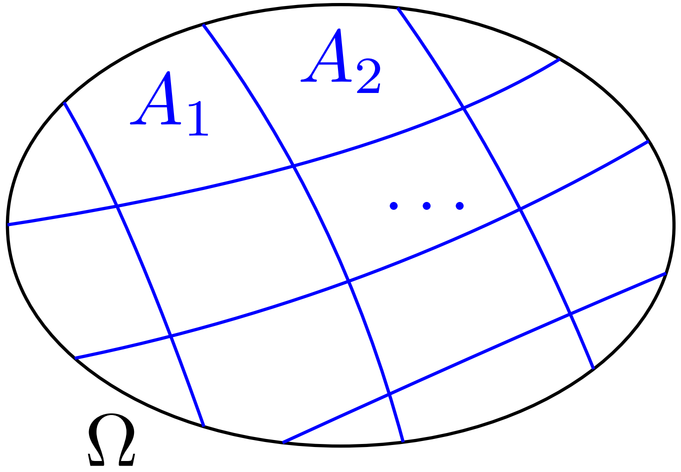
✗ Incompatibility in general

Lemma. If v is smooth, for $A \subseteq \Omega$ starshaped,

$$\mathcal{T}_E(\mu, A) \leq \mathcal{T}_{E,\mathrm{Eul}}(\mu, A) + C|A|\mathrm{diam}(A).$$

Small if A is small

Idea of the proof



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Regularize (μ, v), cut Ω in pieces $A_1, \dots, A_n$ of diameters ε,

$$\overline{\mathcal{T}}_E(\mu, \Omega) \leq \sum_i \mathcal{T}_E(\mu, A_i) \leq \sum_i \mathcal{T}_{E,\mathrm{Eul}}(\mu, A_i) + C\varepsilon m(A_i) \ \leq \mathcal{T}_{E,\mathrm{Eul}}(\mu, \Omega) + C\varepsilon.$$

Conclusion

Question. What is the largest **convex** and **l.s.c.** (for narrow convergence on $\mathcal{M}_+(X \times Y)$) functional $\mathcal{T} : L^0(X, \mathcal{P}(Y), m) \to [0, +\infty]$ such that $\mathcal{T}(\mu_u) = E(u)$ for all u?

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Answers: $\mathcal{T}_{E,\mathrm{Eul}} \leq \mathcal{T}_E$ (for $E(u) = \int W(\nabla u)$)

Eulerian formulation, subadditive envelope

Multimarginal optimal transport

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Open question. Define $E(u) = \displaystyle\int W(\nabla^2 u)$ for W convex. What is the subadditive envelope in this case?

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Thank you for your attention

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