Wasserstein distance between Lévy measures with applications to Bayesian nonparametrics

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Bocconi University

BNP13 - 13th International Conference on Bayesian Nonparametrics, Puerto Varas (Chile), October 27, 2022

Joint work with:

Marta Catalano

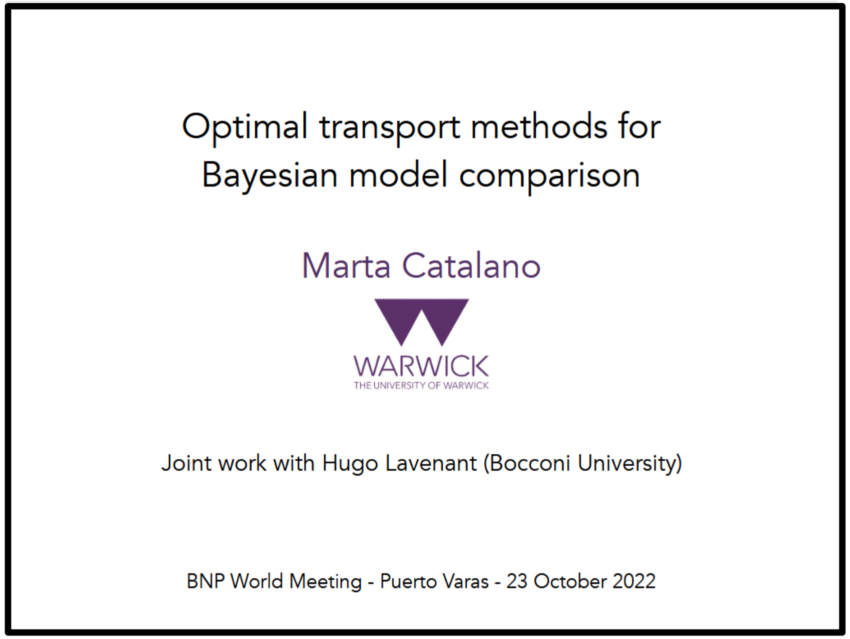
Antonio Lijoi

Igor Prünster

Joint work with:

WARWICK
THE UNIVERSITY OF WARWICK

Marta Catalano



Marta on Tuesday: optimal transport distance between Completely Random Measures to measure the impact of the prior.

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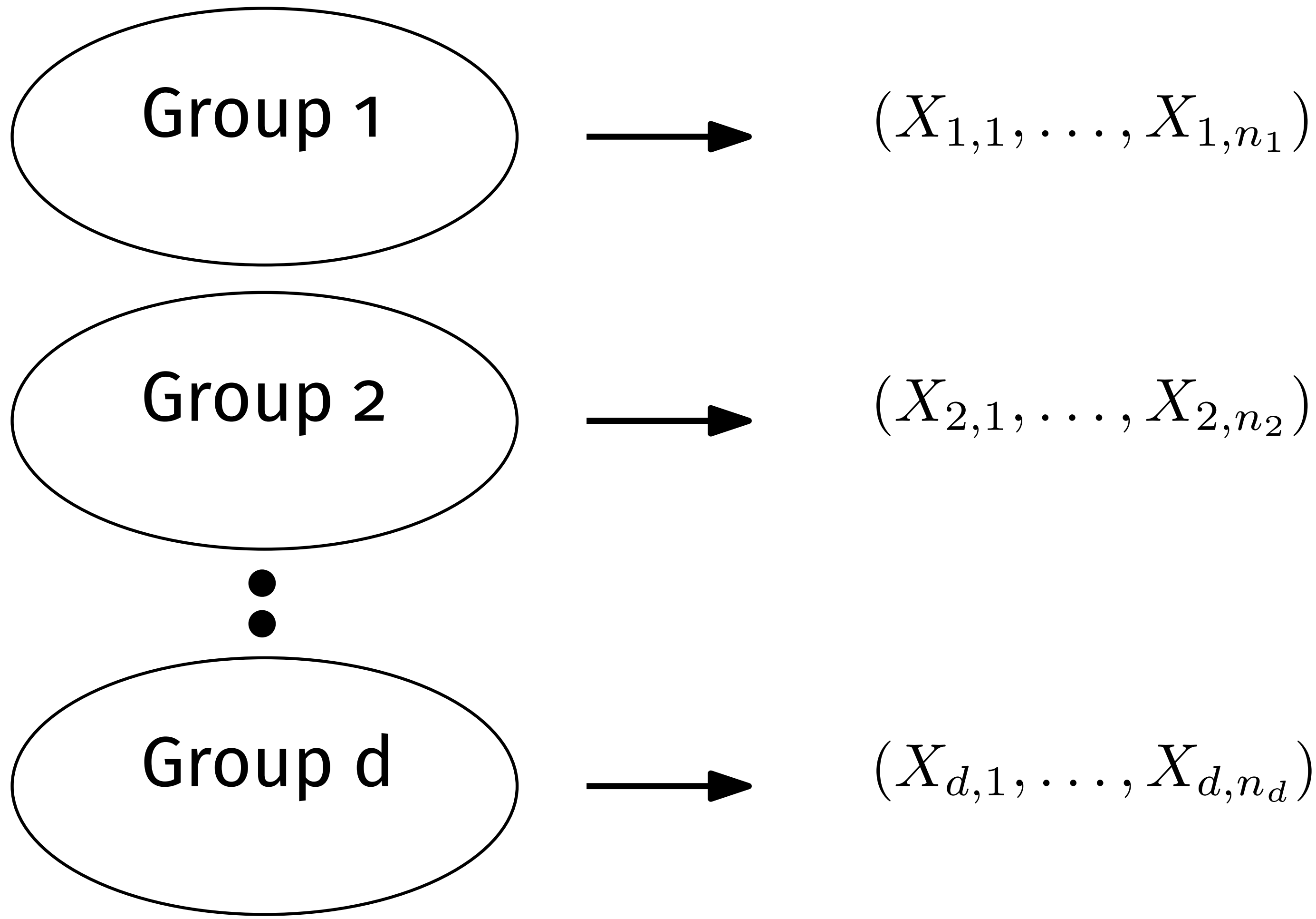
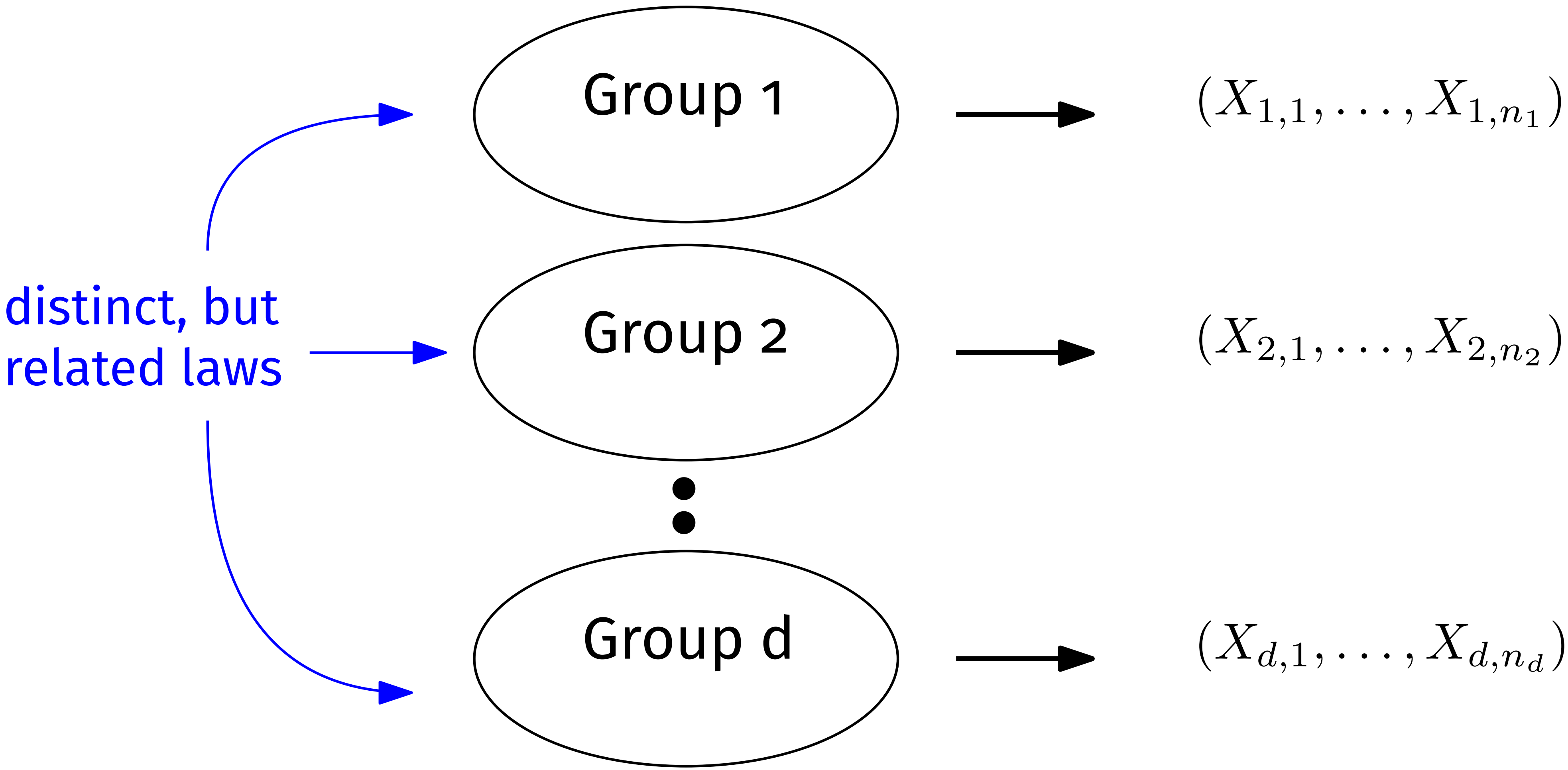
Marta on Tuesday: optimal transport distance between Completely Random Measures to measure the impact of the prior.

Today: optimal transport distance between Completely Random Vectors to measure dependence in the prior.

Quantifying dependence

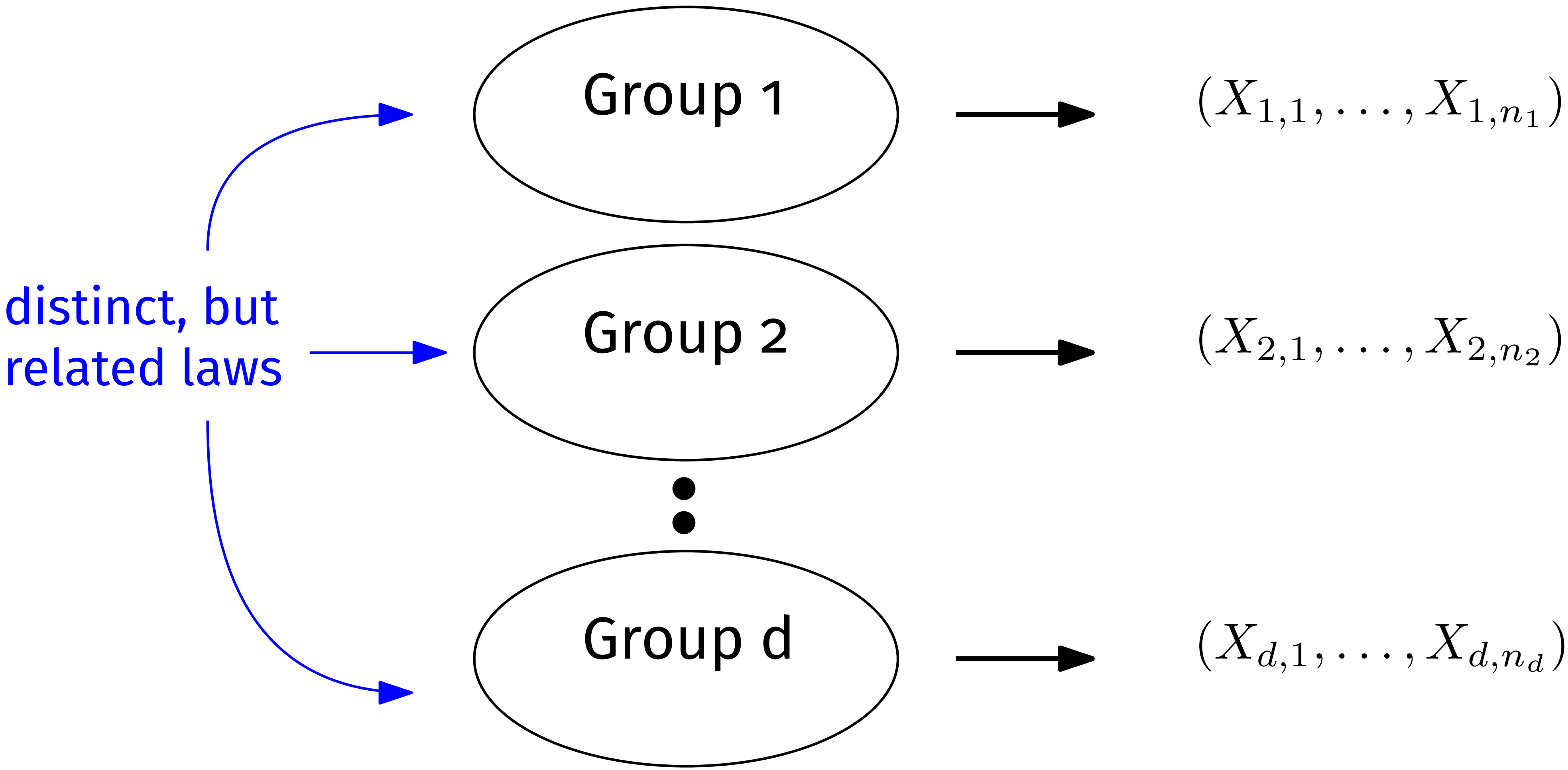


Quantifying dependence



Bayesian inference allows for borrowing of information

Quantifying dependence



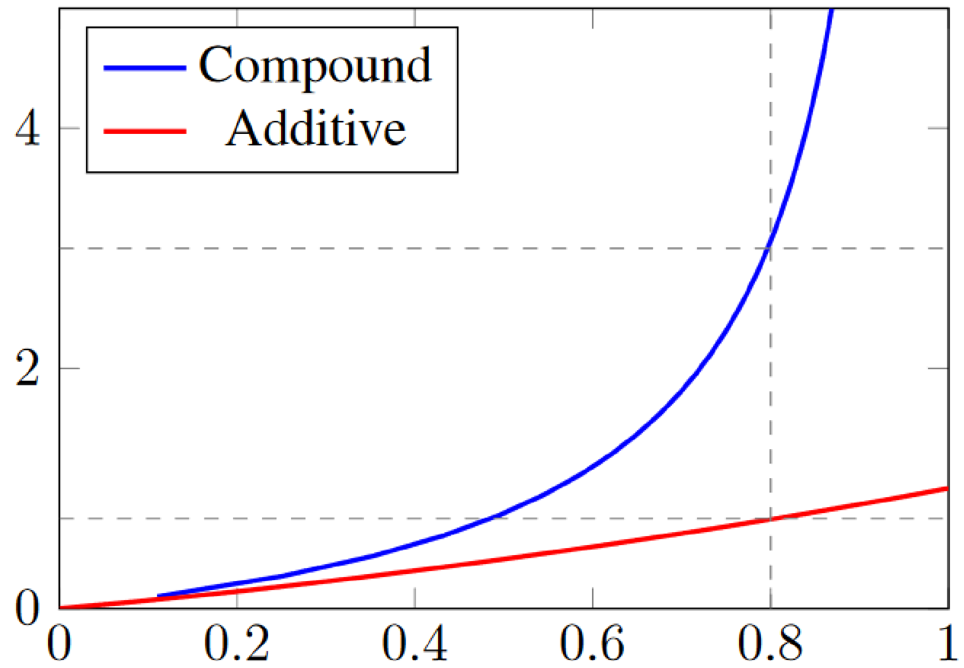
Bayesian inference allows for borrowing of information

Goal: quantifying the amount of **dependence** between groups already present in the **prior**

Snapshot of the final result

Our contribution: an index of dependence quantifying dependence in the prior

Different parametrized models of prior

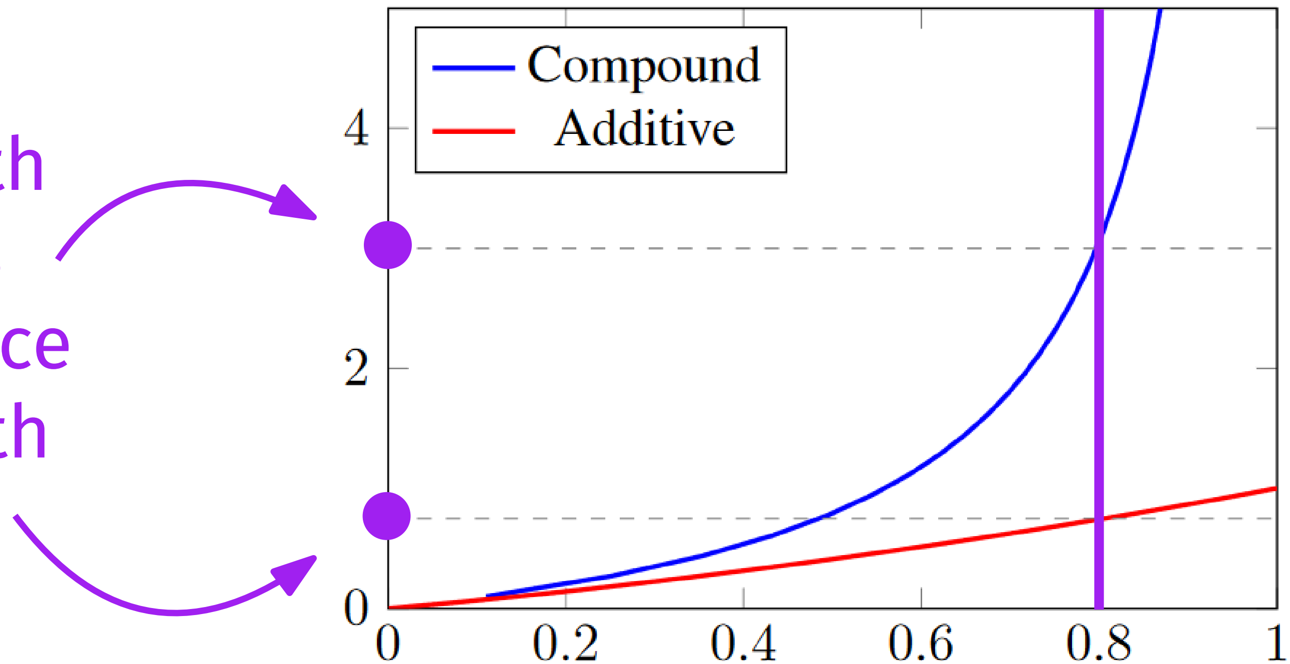


Value of the index

Snapshot of the final result

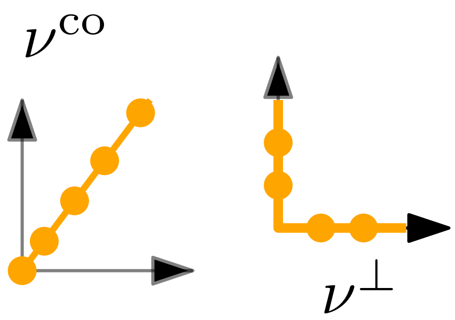
Our contribution: an index of dependence quantifying dependence in the prior

"Compound" with parameter 3 has same dependence as "Additive" with parameter 0.75.



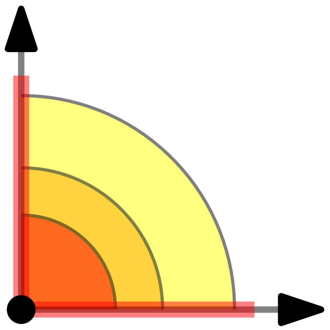
Value of the index

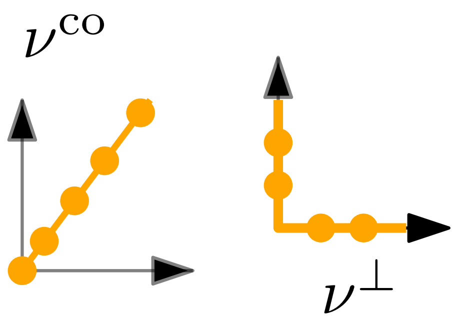
Allow for comparision between **different** priors



1 - Context, general strategy

2 - Building the index with optimal transport





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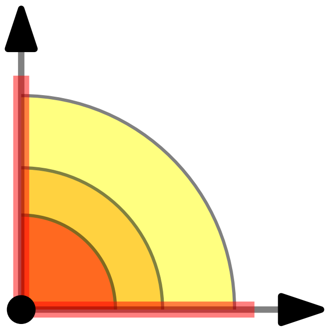
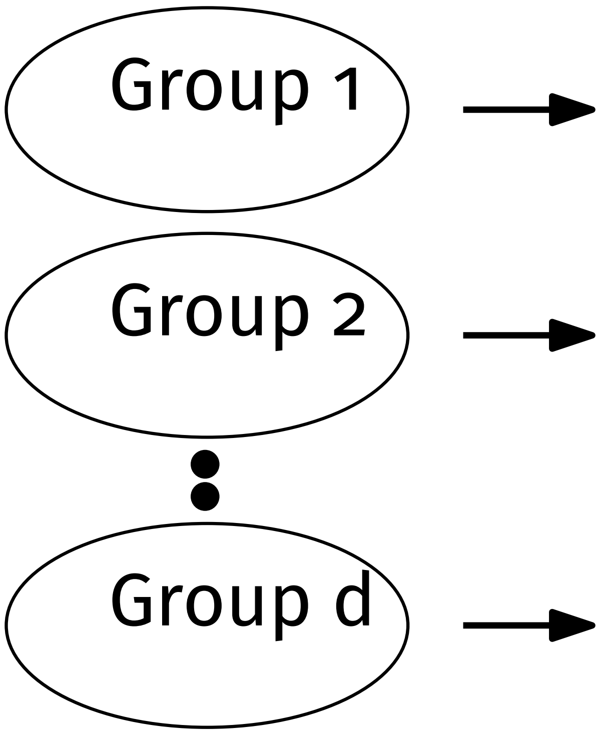


Specific setting: Completely Random Vectors



$$X_{1,1}, X_{1,2}, \dots, X_{1,n_1} \,|\, \tilde{\boldsymbol{\mu}} \overset{\text{i.i.d.}}{\sim} \frac{\tilde{\mu}_1}{\tilde{\mu}_1(\mathbb{X})}$$

$$X_{2,1}, X_{2,2}, \dots, X_{2,n_2} \,|\, \tilde{\boldsymbol{\mu}} \overset{\text{i.i.d.}}{\sim} \frac{\tilde{\mu}_2}{\tilde{\mu}_2(\mathbb{X})}$$

$$\vdots$$

$$X_{d,1}, X_{d,2}, \dots, X_{d,n_d} \,|\, \tilde{\boldsymbol{\mu}} \overset{\text{i.i.d.}}{\sim} \frac{\tilde{\mu}_d}{\tilde{\mu}_d(\mathbb{X})}$$

(justified by partial exchangeability)

Specific setting: Completely Random Vectors

$\tilde{\boldsymbol{\mu}} = (\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_d)$ Completely Random Vector

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Definition (CRV). For all $A_1, \ldots, A_n \subseteq \mathbb{X}$ disjoints, the vectors $\tilde{\boldsymbol{\mu}}(A_1), \ldots, \tilde{\boldsymbol{\mu}}(A_n)$ are independent random vectors in $\mathbb{R}_+^d$.

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Contains all dependence in the prior

$$X_{1,1}, X_{1,2}, \ldots, X_{1,n_1} \mid \tilde{\boldsymbol{\mu}} \overset{\text{i.i.d.}}{\sim} \frac{\tilde{\mu}_1}{\tilde{\mu}_1(\mathbb{X})}$$

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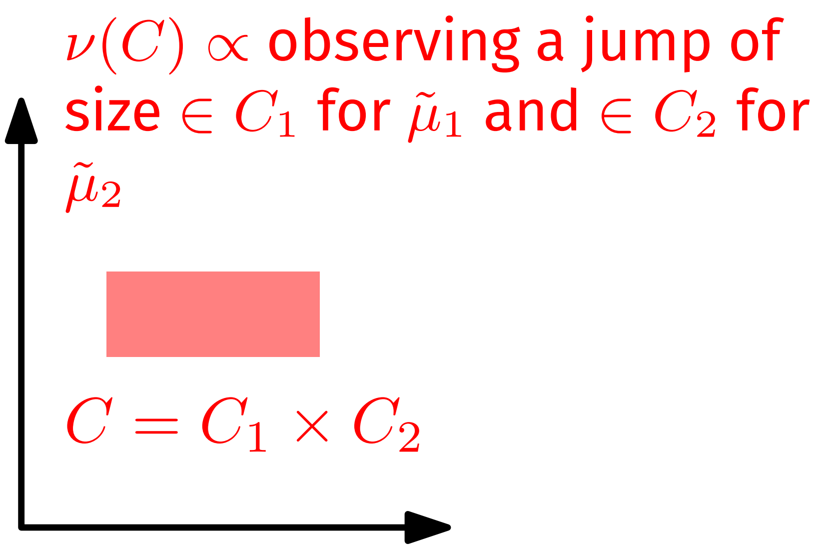
For $A \subseteq \mathbb{X}$, the random variables $\tilde{\mu}_1(A), \ldots, \tilde{\mu}_d(A)$ may be dependent.

Lévy measure of a Completely Random Vector

Assumptions of **homogeneity** and no fixed atoms:

$$\tilde{\boldsymbol{\mu}} = \sum_{i=1}^{\infty} \tilde{\mathbf{J}}_i \delta_{Y_i}$$

where $(Y_i)_i \in \mathbb{X}$ (**atoms**) follow base measure P_0; and $(\tilde{\mathbf{J}}_i)_i$ (**jumps**) independent from $(Y_i)_i$ follow Poisson point cloud on $\mathbb{R}_+^d$ with intensity measure ν (**Lévy measure**).



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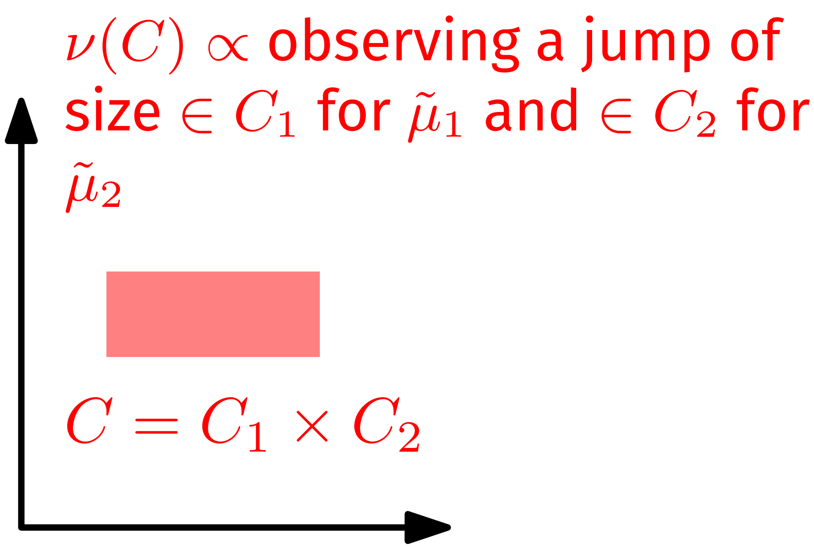

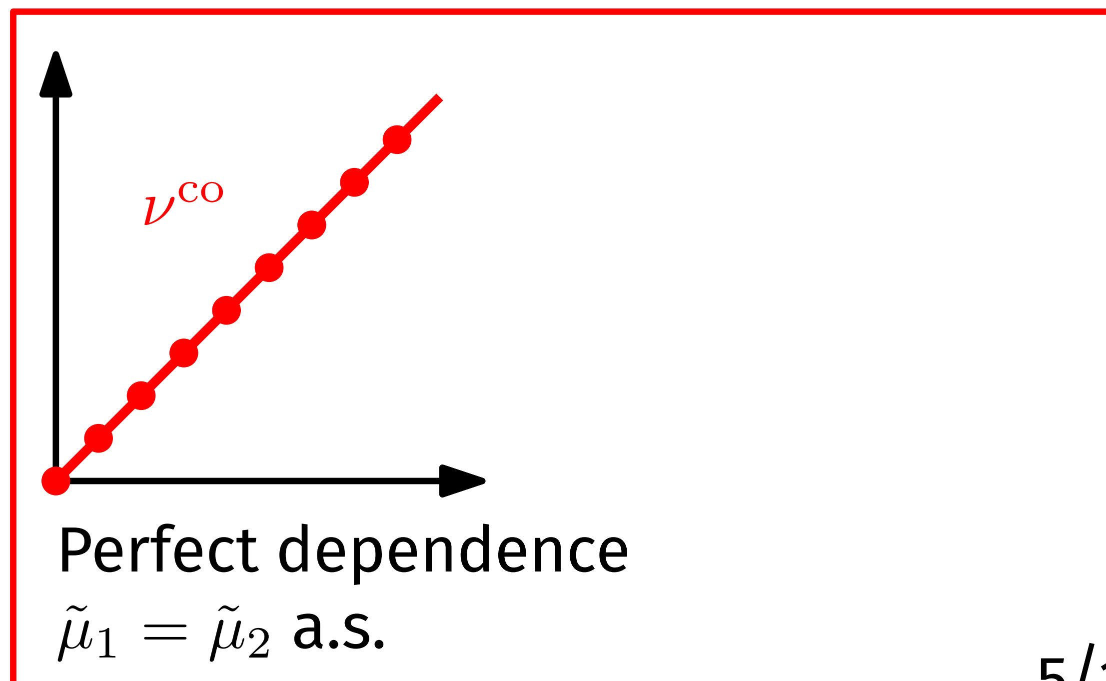

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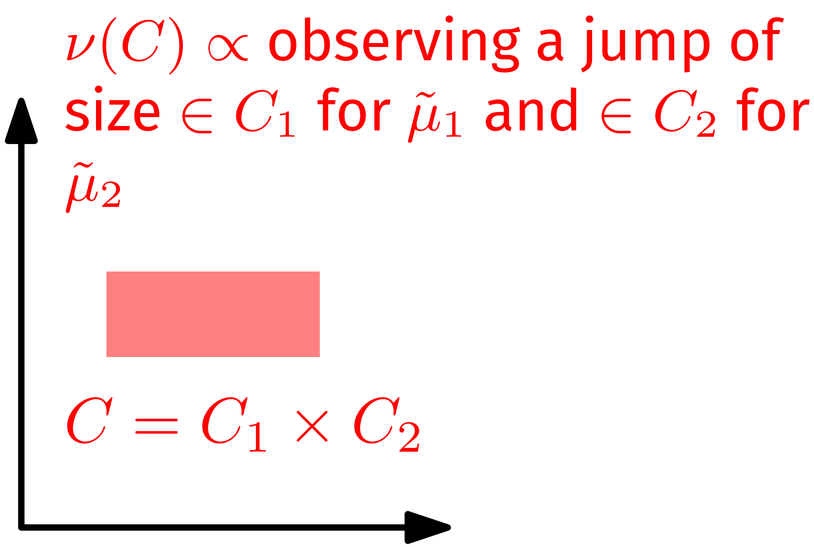
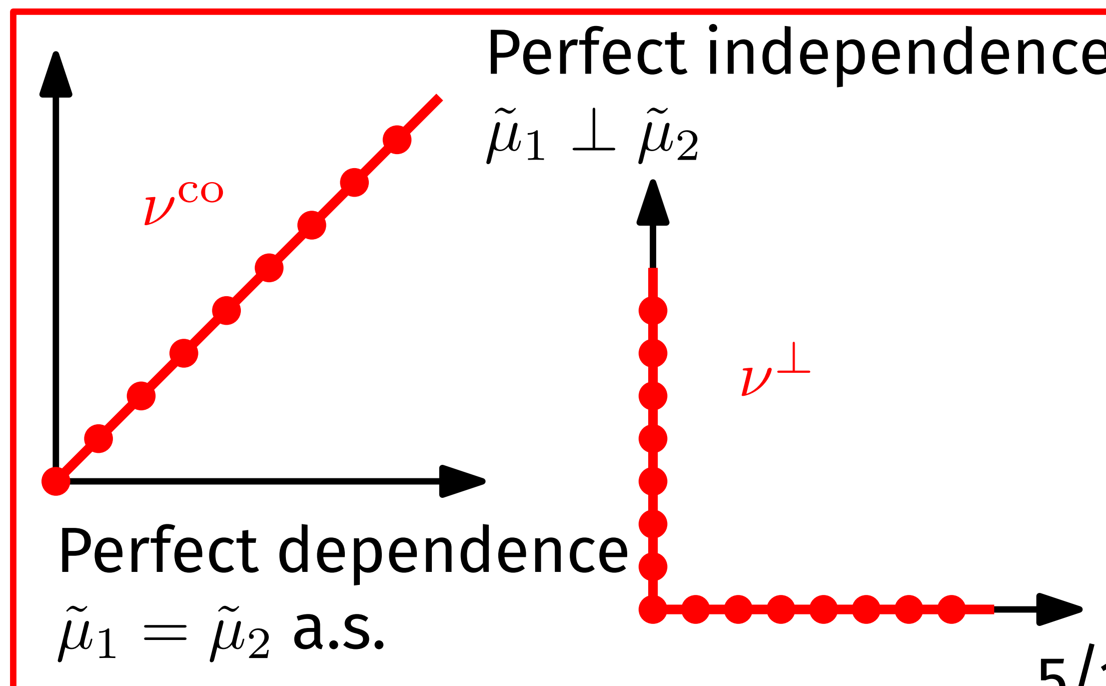
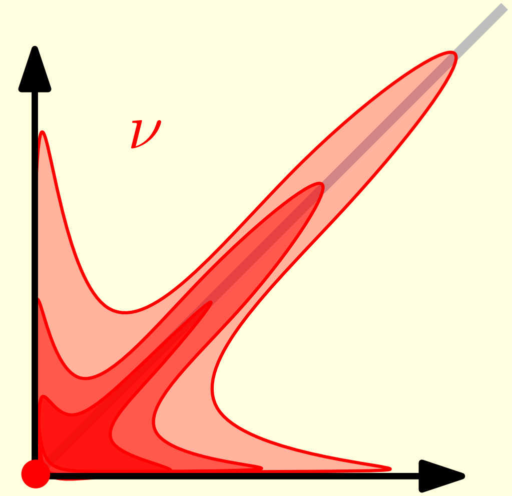




Lévy measure of a Completely Random Vector

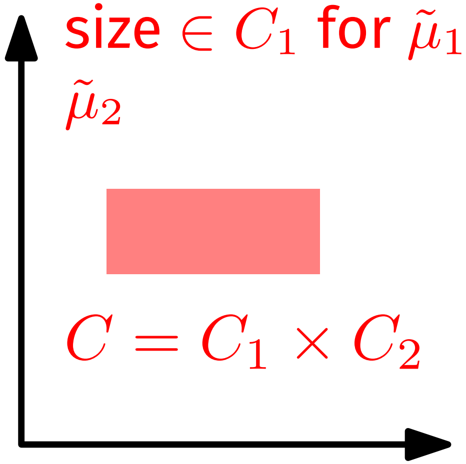
Assumpti

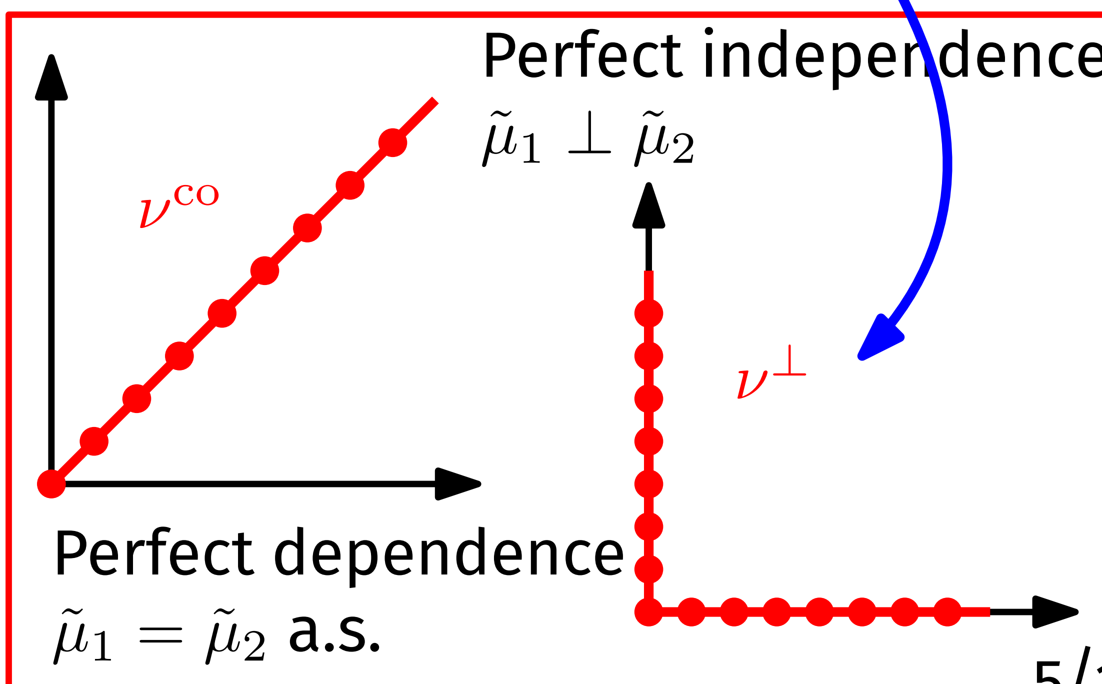
Goal: distinguish between these two cases.



where $(Y$ … $(\tilde{\mathbf{J}}_i)_i$
(jumps) i … cloud
on $\mathbb{R}_+^d$ wi

$\nu(C) \propto$ observing a jump of size $\in C_1$ for $\tilde{\mu}_1$ and $\in C_2$ for $\tilde{\mu}_2$

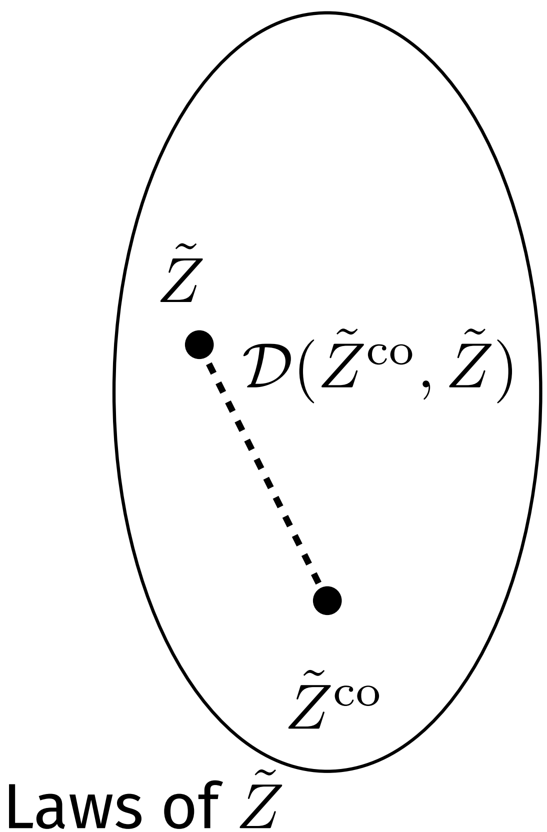




A general method to construct an index

Ingredients:

- $\tilde{Z}$ random object, $\tilde{Z}^{\mathrm{co}}$ "most dependent".
- $\mathcal{D}$ "discrepancy" between laws of random objects.



Móri, and Székely (2020). The Earth Mover's correlation.

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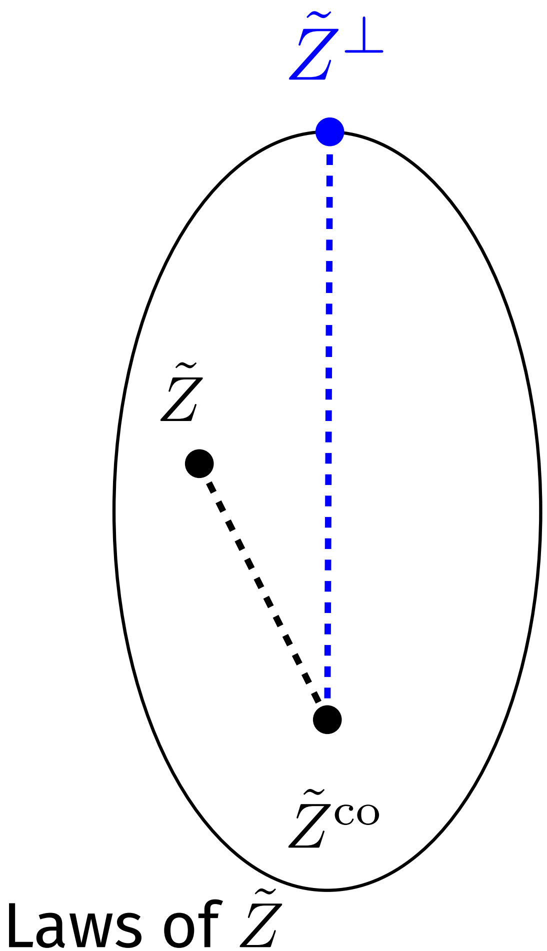

To check: $\mathcal{D}(\tilde{Z}^{\mathrm{co}}, \tilde{Z})$ is maximized when $\tilde{Z} = \tilde{Z}^{\perp}$ the most independent structure.

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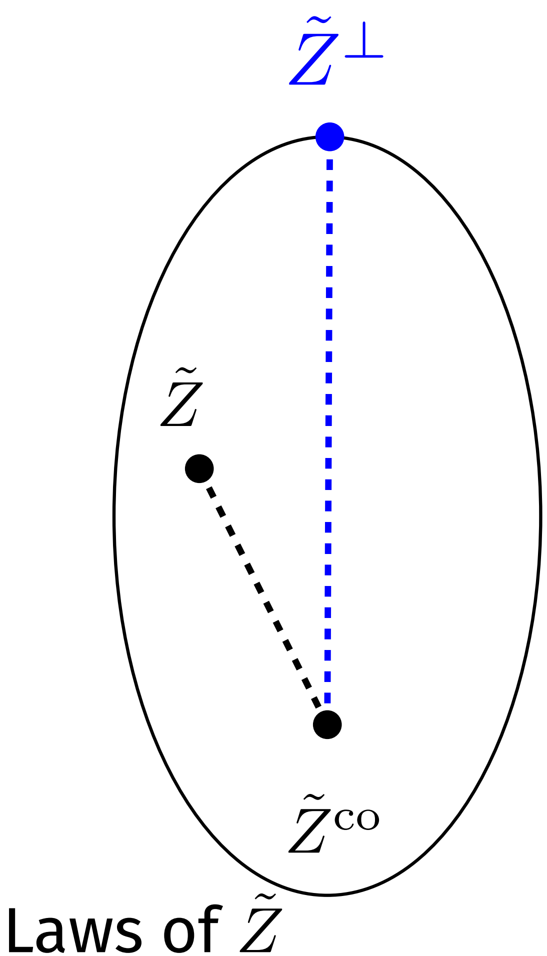

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Then **define**:

$$\mathcal{I}(\tilde{Z}) = 1 - \frac{\mathcal{D}(\tilde{Z}^{\mathrm{co}}, \tilde{Z})}{\mathcal{D}(\tilde{Z}^{\mathrm{co}}, \tilde{Z}^{\perp})}.$$

It belongs to $[0, 1]$ and satisfies:

$$\mathcal{I}(\tilde{Z}^{\perp}) = 0, \qquad \mathcal{I}(\tilde{Z}^{\mathrm{co}}) = 1.$$

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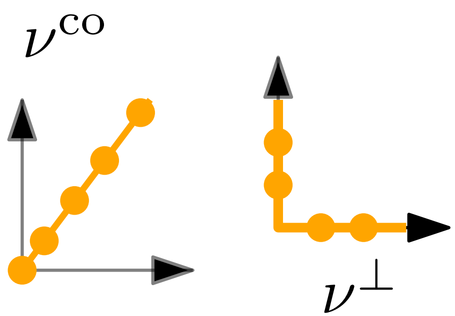

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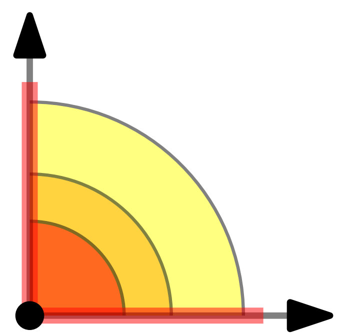

How to measure discrepancy between Completly Random Vectors?

Law of a Completly Random Vector $\tilde{\mu}$

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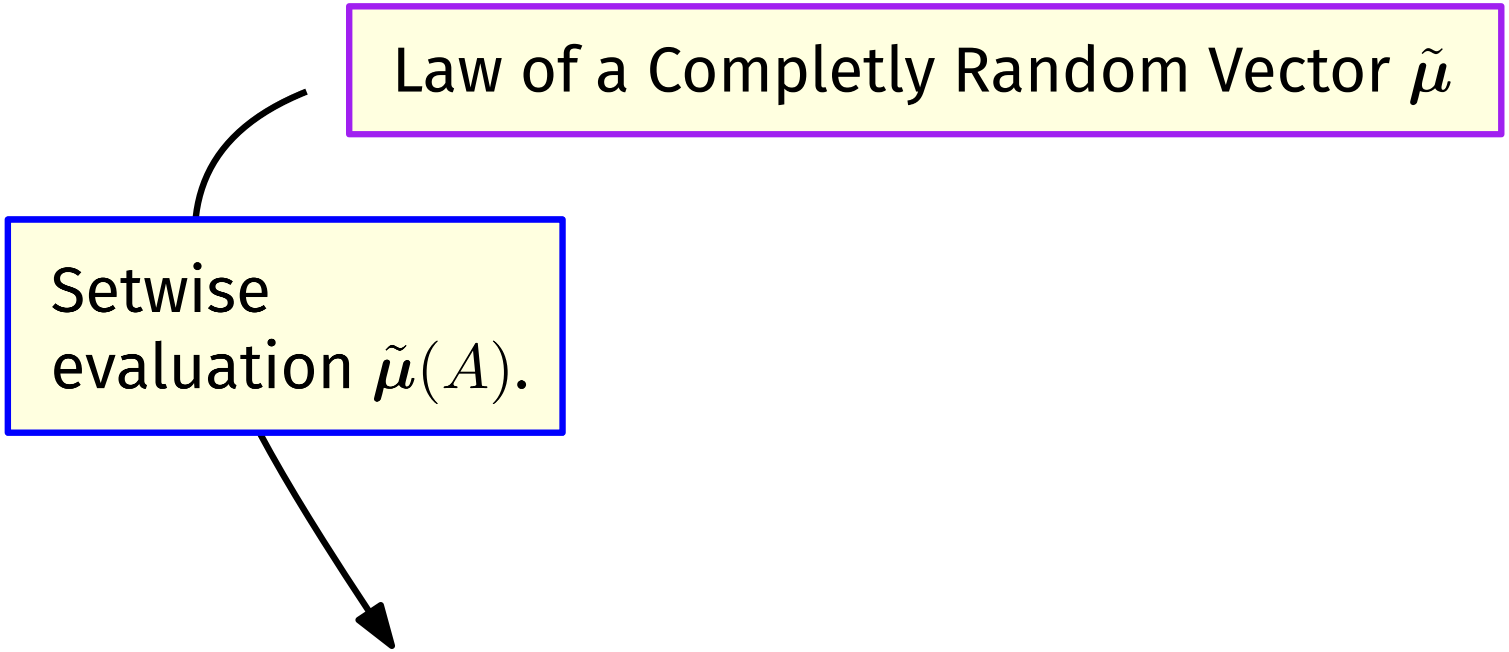

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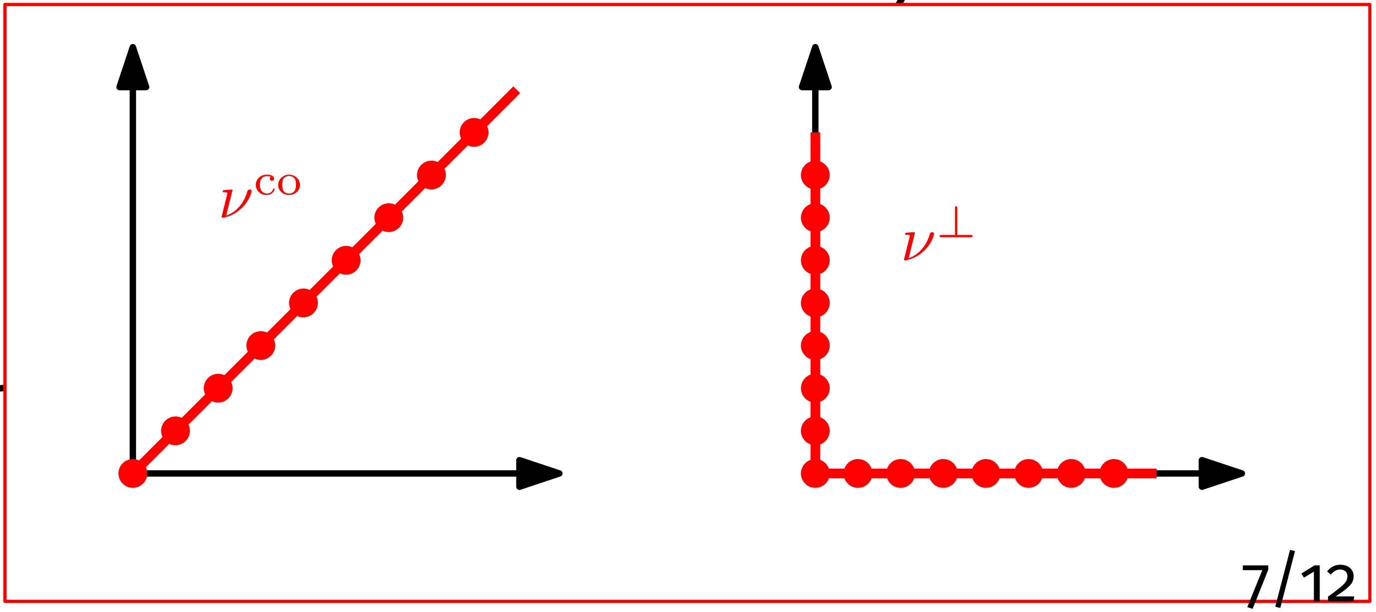

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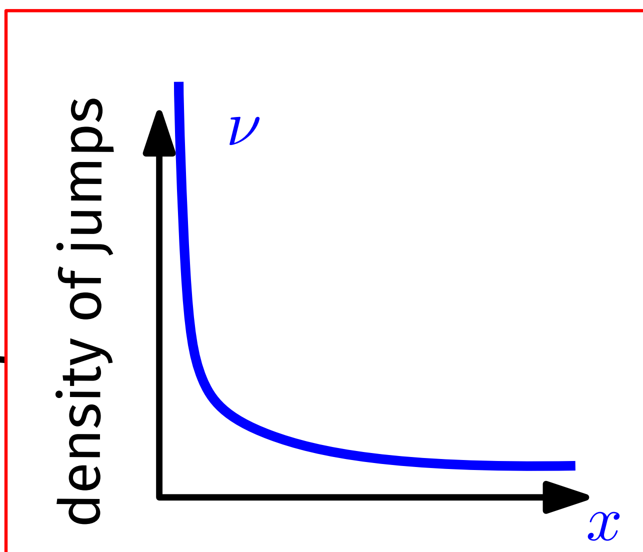
Use Optimal transport

Problem: ν has infinite mass



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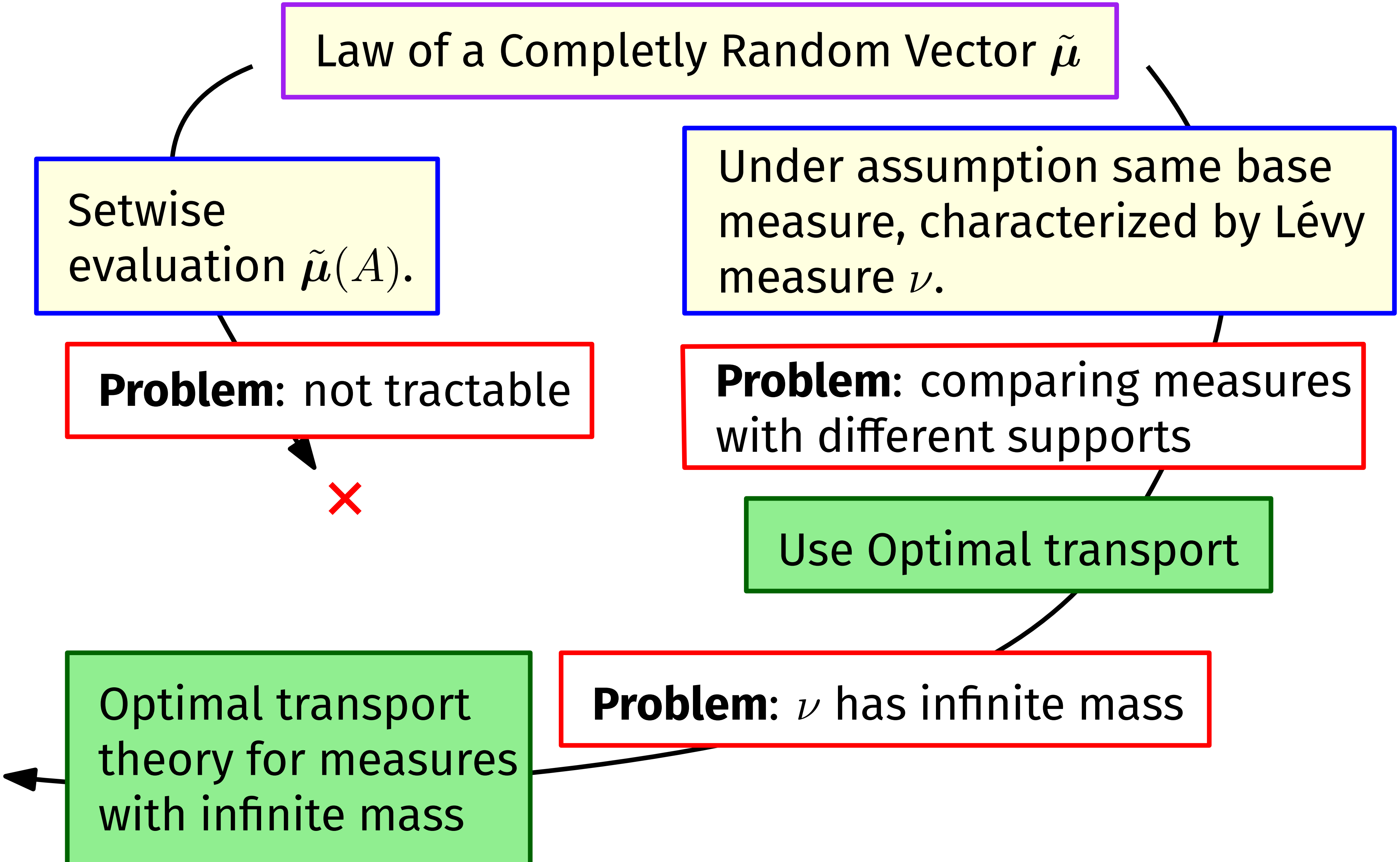

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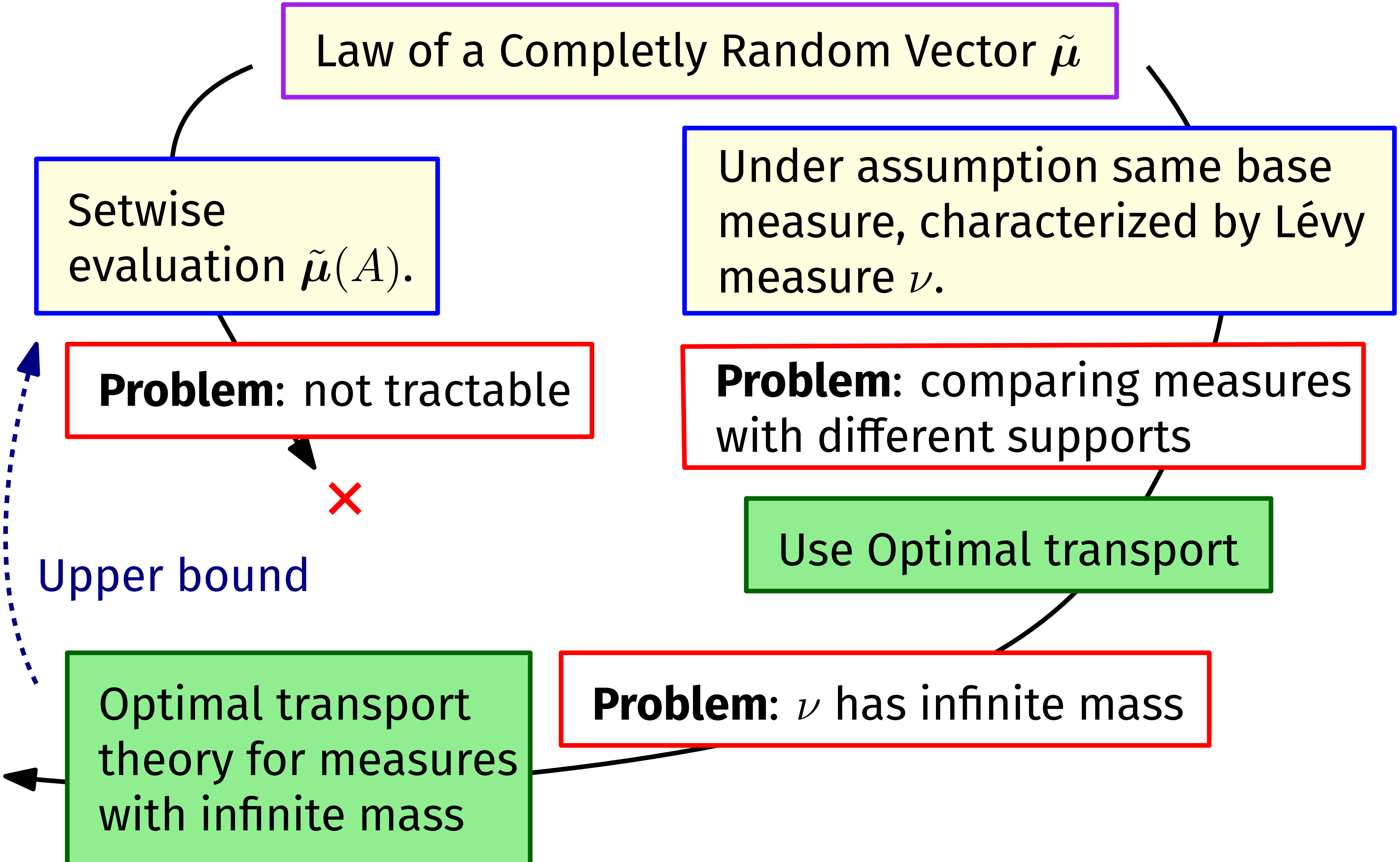

Catalano, Lijoi and Prünster (2021). Measuring dependence in the Wasserstein distance for Bayesian nonparametric models.

(Classical) optimal transport

Definition. If ν^1, ν^2 probability distributions, the Wasserstein distance is

$$\mathcal{W}(\nu^1, \nu^2)^2 = \min_{(X,Y)} \left\{ \mathbb{E}\left[\|X - Y\|^2 \right] \;:\; X \sim \nu^1 \text{ and } Y \sim \nu^2 \right\}$$

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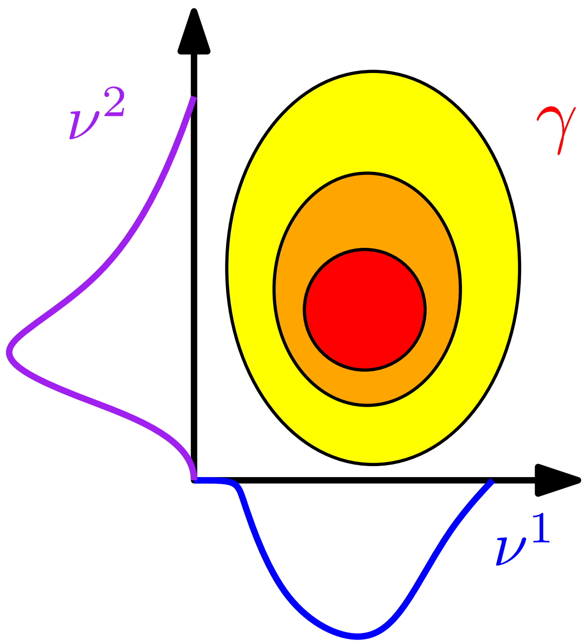
$$= \min_{\gamma} \left\{ \iint \|x-y\|^2 \mathrm{d}\gamma(x,y) \ : \ \pi_1 \# \gamma = \nu^1 \text{ and } \pi_2 \# \gamma = \nu^2 \right\}$$



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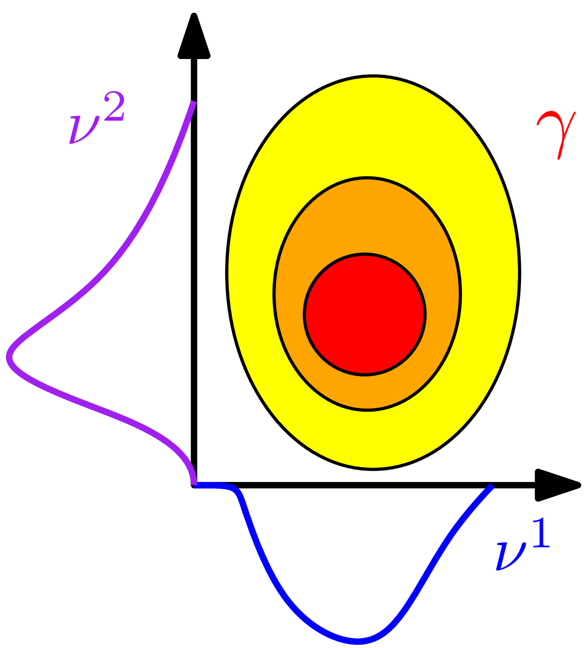
$$\leq \int \|x\|^2 \mathrm{d}\nu^1(x) + \int \|y\|^2 \mathrm{d}\nu^2(y)$$



Observation. Naively, makes sense if ν^1, ν^2 have infinite mass but **finite** second moment.

Extended Wasserstein distance

Definition. If ν^1, ν^2 positive measures on $\mathbb{R}^d_+ \setminus \{0\}$ with **finite second moments**, the Wasserstein distance is

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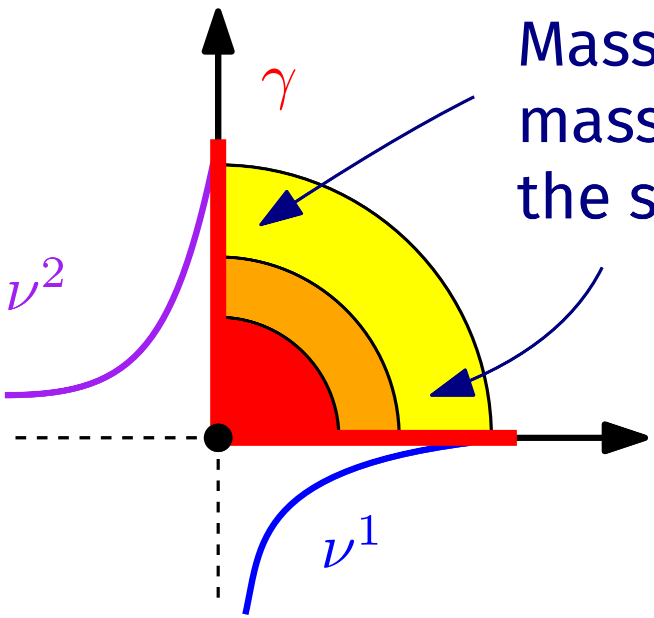
with γ measure on $\mathbb{R}^{2d}_+ \setminus \{(0,0)\}$.



Mass on $\mathbb{R}^d_{+,*} \times \{0\}$ and $\{0\} \times \mathbb{R}^d_{+,*}$: mass “destroyed” or “created” from the sink/reservoir $(0,0)$.

Figalli and Gigli (2010). A new transportation distance between non-negative measures, with applications to gradients flows with Dirichlet boundary conditions.

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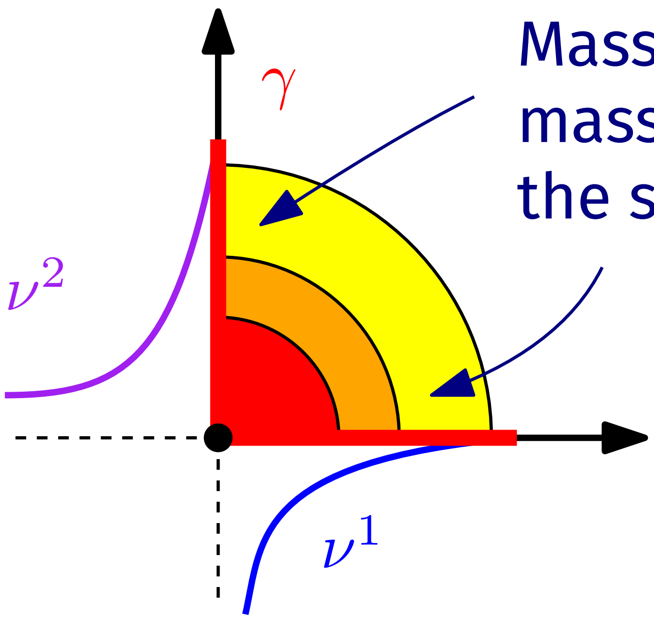

Mass on $\mathbb{R}_{+,*}^d \times \{0\}$ and $\{0\} \times \mathbb{R}_{+,*}^d$: mass “destroyed” or “created” from the sink/reservoir $(0,0)$.

Marta’s talk: couple also the law of atoms for inhomogeneous CRM. Used to quantify impact of the prior.

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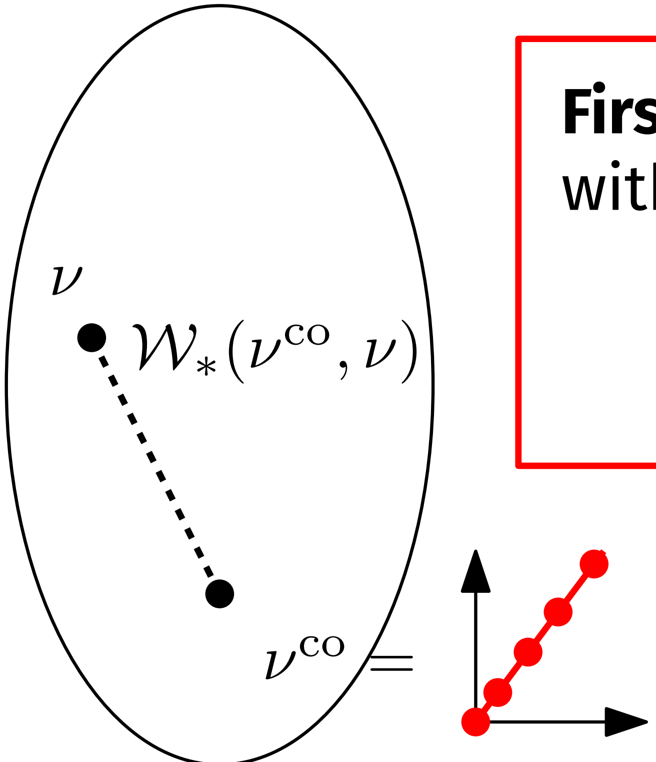
Building the index

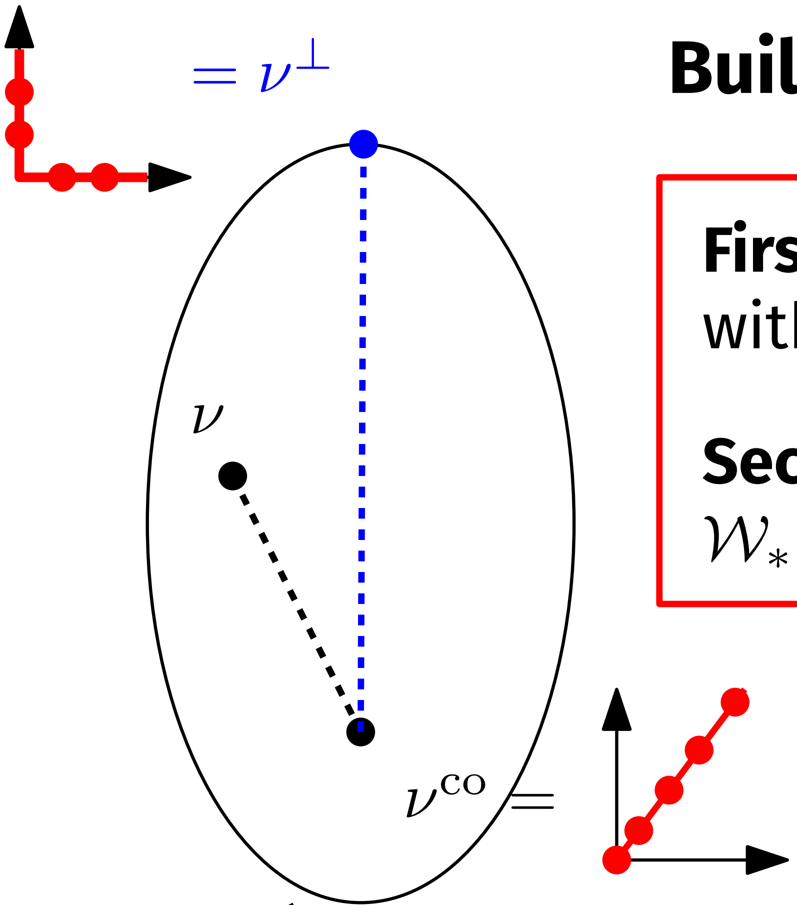


Space of Lévy measure over $\mathbb{R}_+^d$ having same marginals

First result. $\mathcal{W}_*(\nu^{\mathrm{co}}, \nu)$ can be computed with 1d integrals of tail functions.

Catalano, Lavenant, Lijoi, Prünster (2022+). A Wasserstein index of dependence.

Building the index



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First result. $\mathcal{W}_*(\nu^{\text{co}}, \nu)$ can be computed with 1d integrals of tail functions.

Second result. If ν^{co} has infinite mass, $\mathcal{W}_*(\nu^{\text{co}}, \nu)$ is maximized exactly for $\nu = \nu^{\perp}$.

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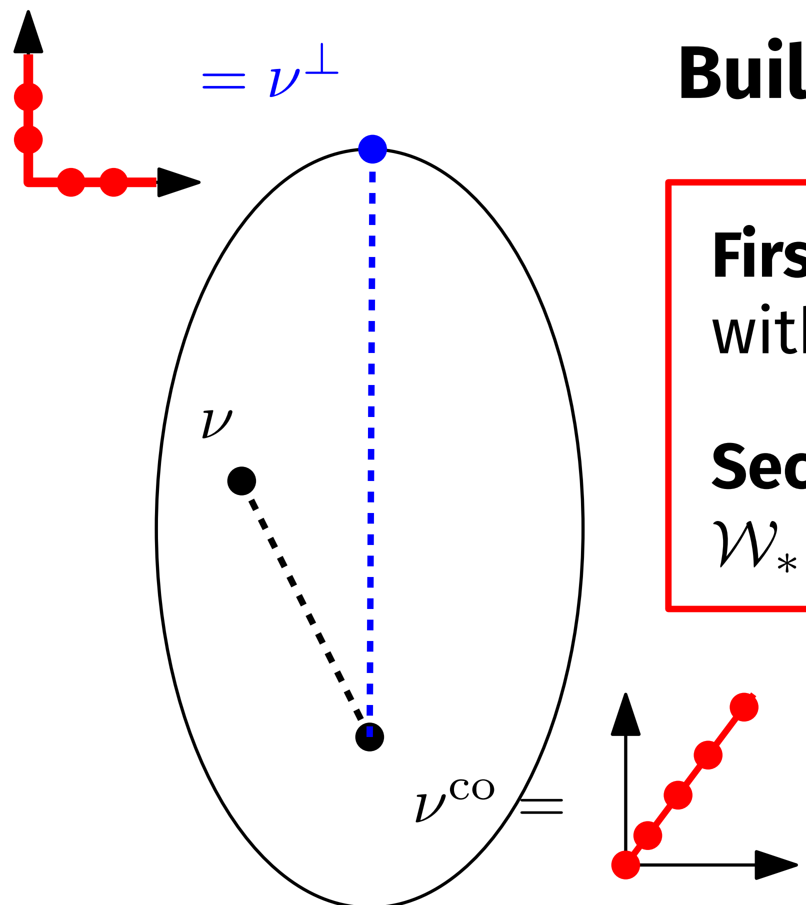

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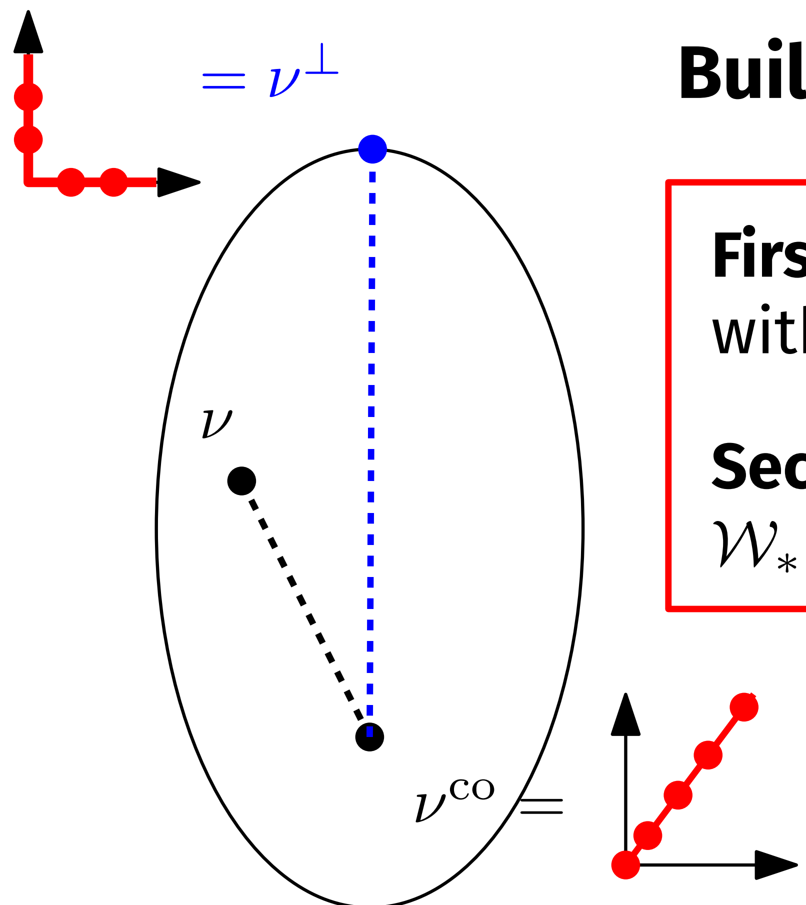
Building the index



Space of Lévy measure over $\mathbb{R}_+^d$ having same marginals

First result. $\mathcal{W}_*(\nu^{\mathrm{co}}, \nu)$ can be computed with 1d integrals of tail functions.

Second result. If ν^{co} has infinite mass, $\mathcal{W}_*(\nu^{\mathrm{co}}, \nu)$ is maximized exactly for $\nu = \nu^{\perp}$.

Define:

$$\mathcal{I}(\nu) = 1 - \frac{\mathcal{W}_*(\nu^{\mathrm{co}}, \nu)^2}{\mathcal{W}_*(\nu^{\mathrm{co}}, \nu^{\perp})^2}.$$

It belongs to $[0, 1]$ and satisfies:

$$\mathcal{I}(\nu^{\perp}) = 0, \qquad \mathcal{I}(\nu^{\mathrm{co}}) = 1.$$

Consequence. We have an index of dependence for homogeneous infinitely active completely random vectors without fixed atoms, with equal marginals and finite second moments.

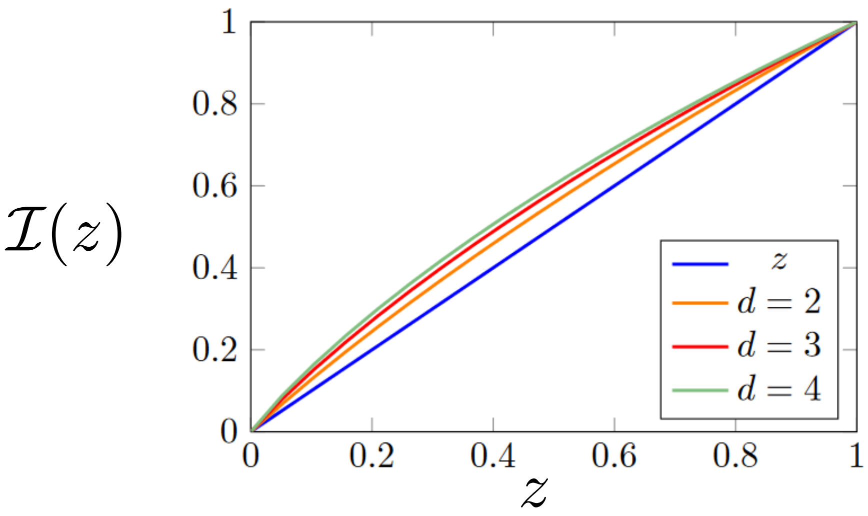
Catalano, Lavenant, Lijoi, Prünster (2022+). A Wasserstein index of dependence.

Examples

Additive model

Parameter $z \in [0, 1]$,

$$\nu = (1 - z)\nu^{\perp} + z\nu^{\text{co}}$$



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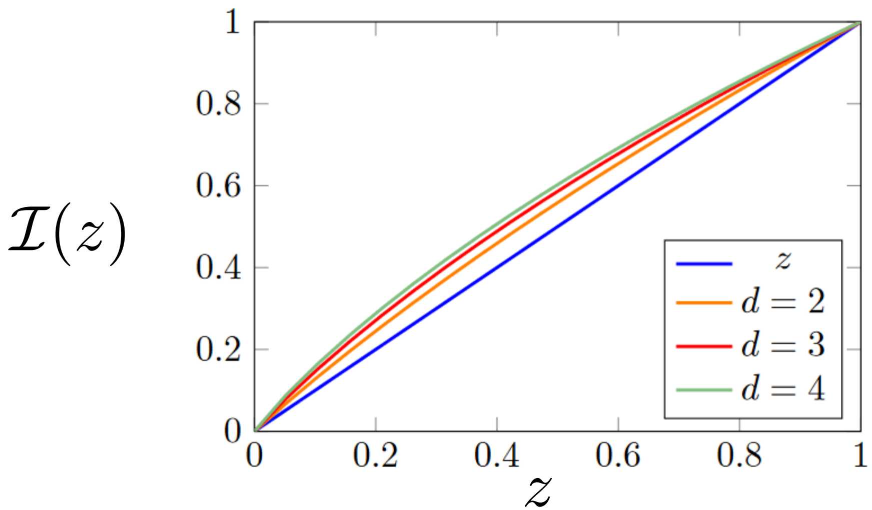

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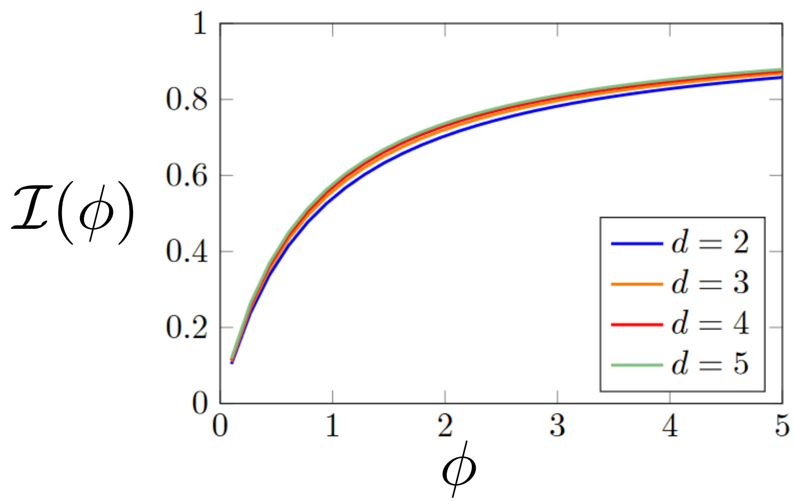
Compound random measures

Parameter ϕ measures dependence

$$\nu(s_1, \ldots, s_d) = \int_0^{+\infty} h^{\phi}\left(\frac{s_1}{u}, \ldots, \frac{s_d}{u}\right) \mathrm{d}\nu_*^{\phi}(u)$$

for well chosen h^{ϕ}, ν_*^{ϕ}.



Lijoi, Nipoti and Prünster (2014). Bayesian inference with dependent normalized completely random measures.
Griffin and Leisen (2017). Compound random measures and their use in bayesian non-parametrics.

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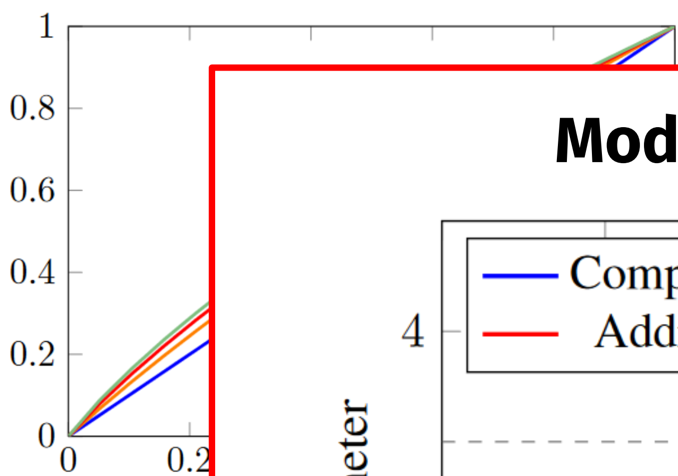
$\mathcal{I}(z)$



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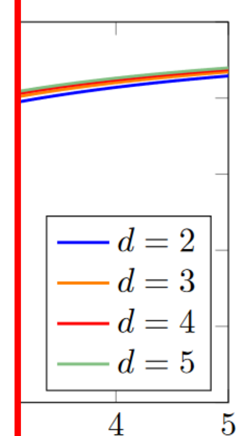
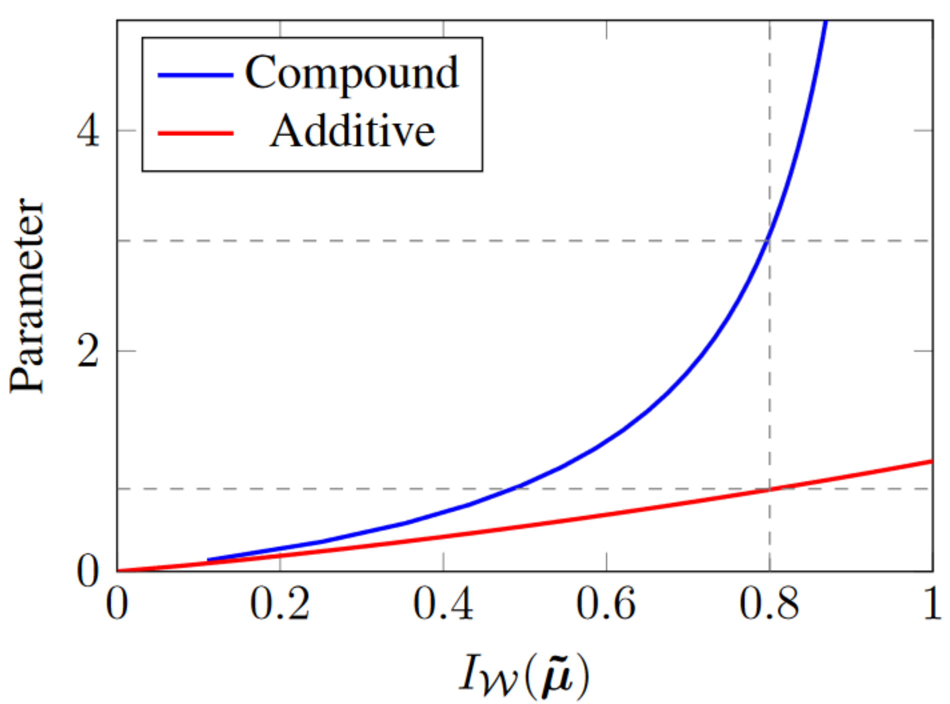


Model comparision



Lijoi, Nipoti and Prünste

Griffin and Leisen (2017)

Conclusion

What is done:

- Wasserstein distance between Lévy measures.
- Index of dependence between Completely Random Vectors.

What's next?:

- Study dependence in the posterior.
- Use this distance for other purposes: measuring the impact of the prior, hypothesis testing, etc.

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Thank you for your attention