Harmonic mappings valued in the Wasserstein space

Hugo Lavenant[a]

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Diff. Geom, Math. Phys., PDE Seminar – University of British Columbia

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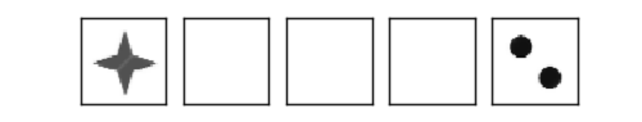
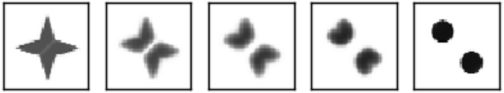
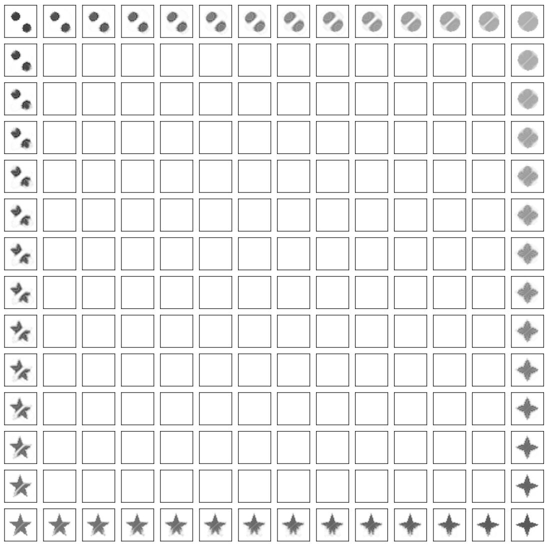
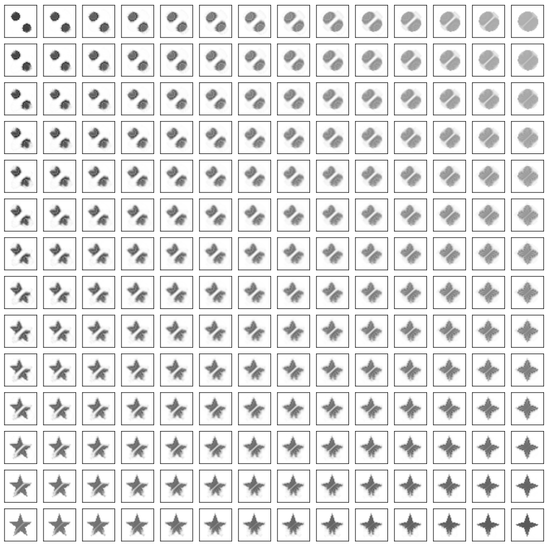






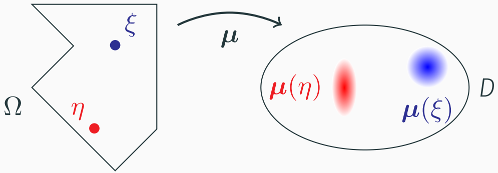


Measure-valued mappings

Ω bounded set of $\mathbb{R}^n$ with Lipschitz boundary, D bounded convex set of $\mathbb{R}^d$.

$\mathcal{P}(D)$ set of probability measures on D, "Wasserstein space".

We study $\boldsymbol{\mu} : \Omega \to \mathcal{P}(D)$.



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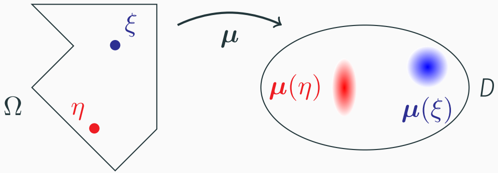

Definition of $\mathrm{Dir}(\boldsymbol{\mu}) = \dfrac{1}{2}\displaystyle\int_\Omega |\nabla \boldsymbol{\mu}|^2_{W_2}$ the **Dirichlet energy** w.r.t. quadratic Wasserstein distance.

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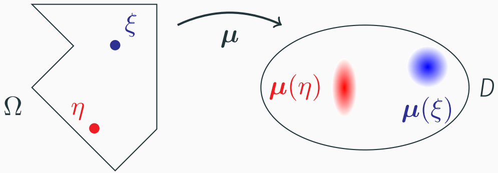

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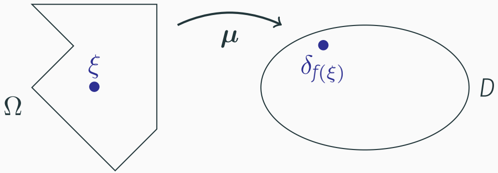

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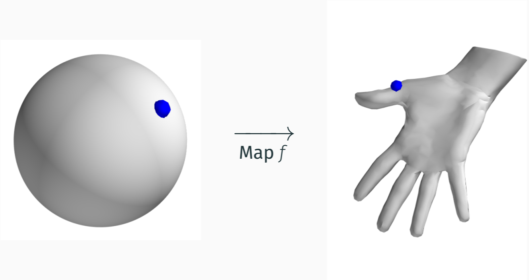
If $f : \Omega \to D$ and $\boldsymbol{\mu}(\xi) := \delta_{f(\xi)}$ then $\mathrm{Dir}(\boldsymbol{\mu}) = \dfrac{1}{2} \displaystyle\int_\Omega |\nabla f|^2$.

Surface mapping (Solomon et. al.)

$\mathcal{M}$ and $\mathcal{N}$ are surfaces embedded in $\mathbb{R}^3$.

We want a map $f : \mathcal{M} \to \mathcal{N}$ with the least stretching.



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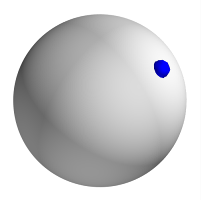
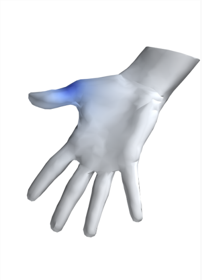
The constraint of being single valued is relaxed by taking $\boldsymbol{\mu} : \mathcal{M} \to \mathcal{P}(\mathcal{N})$. The problem becomes *convex*.

$\mathrm{Dir}(\boldsymbol{\mu}) = \int_{\mathcal{M}} \frac{1}{2} |\nabla \boldsymbol{\mu}|^2$ is a measure of the stretching of $\boldsymbol{\mu}$.



$\xrightarrow[\text{Map } \boldsymbol{\mu}]{}$



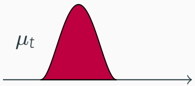
In this presentation

1. The Wasserstein space and the Dirichlet energy
2. The Dirichlet problem
3. What can be said about these harmonic mappings?

1. The Wasserstein space and the Dirichlet energy

The metric tensor in the Wasserstein space

$D \subset \mathbb{R}^d$ bounded convex, $\mathcal{P}(D)$ is the “Wasserstein space”.



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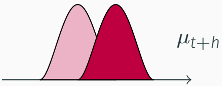

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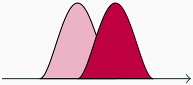
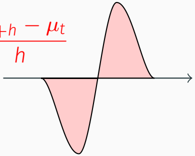


Vertical derivative

$\frac{\mu_{t+h} - \mu_t}{h}$



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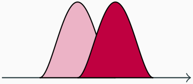

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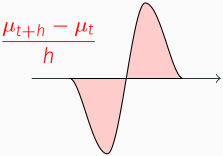
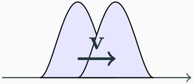


Horizontal derivative



A particle located at x moves to $x + h\mathbf{v}(x)$

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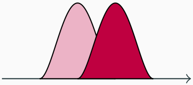

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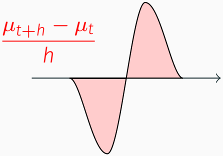
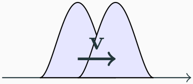


$$\boxed{-\partial_t \mu = \nabla \cdot (\mu \mathbf{v})}$$

Horizontal derivative



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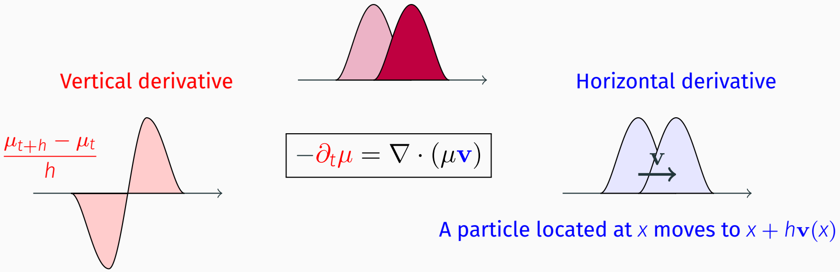

- Quadratic Optimal Transport: the square of the norm of the speed is

$$\min_{\mathbf{v}:D\to\mathbb{R}^d} \left\{ \int_D |\mathbf{v}(x)|^2 \, \mu(\mathrm{d}x) \ : \ \nabla \cdot (\mu \mathbf{v}) = -\partial_t \mu \right\}.$$

Curves valued in the Wasserstein space

If $\boldsymbol{\mu} : [0,1] \to \mathcal{P}(D)$ is given, its Dirichlet energy (or **action**) is

$$\mathrm{Dir}(\boldsymbol{\mu}) := \min_{\mathbf{v}} \left\{ \frac{1}{2} \int_0^1 \int_D |\mathbf{v}|^2 \, \mathrm{d}\boldsymbol{\mu} \, \mathrm{d}t \ : \ \partial_t \boldsymbol{\mu} + \nabla \cdot (\boldsymbol{\mu} \mathbf{v}) = 0 \right\}.$$

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The Wasserstein distance W_2 is

$$\frac{1}{2} W_2^2(\rho, \nu) = \min_{\boldsymbol{\mu}} \left\{ \mathrm{Dir}(\boldsymbol{\mu}) \ : \ \boldsymbol{\mu}_0 = \rho, \ \boldsymbol{\mu}_1 = \nu \right\},$$

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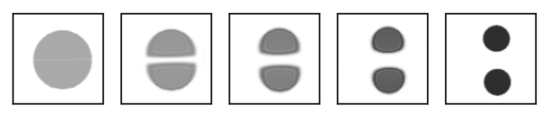

and the minimizers are the constant-speed geodesics.

The Dirichlet energy

Definition (Brenier (2003))

If $\boldsymbol{\mu} : \Omega \to \mathcal{P}(D)$ is given,

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where $\mathbf{v} : \Omega \times D \to \mathbb{R}^{nd}$.

If $\Omega = [0,1]$ it coincides with the previous definition.

A definition in arbitrary metric space

If $f : \Omega \to \mathbb{R}$ is smooth,

$$\mathrm{Dir}(f) = \int_\Omega |\nabla f(\xi)|^2 \mathrm{d}\xi =$$

$$\frac{|f(\xi) - f(\eta)|^2}{\varepsilon^2}$$

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Definition (Korevaar and Schoen (1993), Jost (1994))

If (X, δ) is a separable metric space and $f : \Omega \to X$, then

$$\mathrm{Dir}_\varepsilon(f) = \frac{C_n}{2\varepsilon^{n+2}} \iint_{\Omega \times \Omega} \delta(f(\xi), f(\eta))^2 \mathbb{1}_{|\xi - \eta| \leqslant \varepsilon} \mathrm{d}\xi \mathrm{d}\eta.$$

The Dirichlet energy of f is then defined as the limit of $\mathrm{Dir}_\varepsilon(f)$ when ε goes to 0.

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Defined in arbitrary metric spaces but the analysis can be carried out only for spaces of **NonPositive Curvature**.

$(\mathcal{P}(D), W_2)$ has a **positive** curvature.

Equivalence of the definitions

If $\boldsymbol{\mu} : \Omega \to \mathcal{P}(D)$, one sets

$$\mathrm{Dir}_\varepsilon(\boldsymbol{\mu}) := \frac{C_n}{2\varepsilon^{n+2}} \iint_{\Omega\times\Omega} W_2^2(\boldsymbol{\mu}(\xi), \boldsymbol{\mu}(\eta)) \mathbb{1}_{|\xi-\eta|\leqslant\varepsilon} \mathrm{d}\xi \mathrm{d}\eta.$$

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Theorem

One has

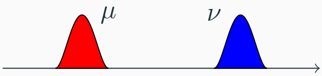
$$\lim_{\varepsilon\to 0} \mathrm{Dir}_\varepsilon = \mathrm{Dir},$$

and the convergence holds pointwisely and in the sense of Γ-convergence along the sequence $\varepsilon_m = 2^{-m}$.

The space $\{\boldsymbol{\mu} \ : \ \mathrm{Dir}(\boldsymbol{\mu}) < +\infty\}$ coincides with $H^1(\Omega, \mathcal{P}(D))$ for the standard definitions of Sobolev spaces in metric spaces (Reshetnyak, Hajłasz).

Curvature and convexity

If $\mu, \nu \in \mathcal{P}(D)$, two ways to interpolate.



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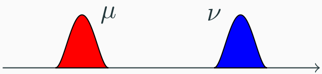

The **displacement** interpolation

- Midpoint of the geodesic in the Wasserstein space.
- The space $(\mathcal{P}(D), W_2)$ is a **positively curved space**: no convexity of W_2^2 nor Dir.

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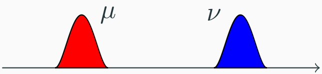

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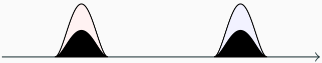
The **linear** interpolation



- The Wasserstein distance square W_2^2 and the Dirichlet energy are convex.
- Tools from convex analysis.

2. The Dirichlet problem

The Dirichlet problem

We choose $\boldsymbol{\mu}_b : \partial\Omega \to \mathcal{P}(D)$ the boundary data.

Definition

The Dirichlet problem is

$$\min_{\boldsymbol{\mu}} \left\{ \mathrm{Dir}(\boldsymbol{\mu}) \ : \ \boldsymbol{\mu} = \boldsymbol{\mu}_b \text{ on } \partial\Omega \right\}.$$

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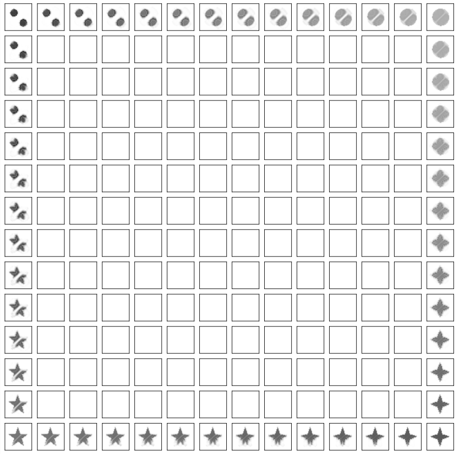
Theorem

Assume $\boldsymbol{\mu}_b : \partial\Omega \to (\mathcal{P}(D), W_2)$ is a Lipschitz mapping. Then there exists at least one solution to the Dirichlet problem.

Tool: extension theorem for Lipschitz mappings valued in $(\mathcal{P}(D), W_2)$. $\square$

Uniqueness is an open question.

Numerics: example



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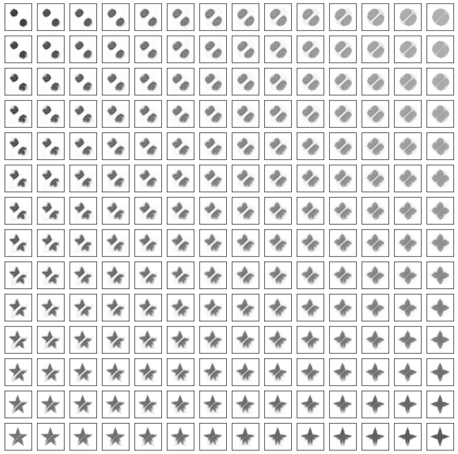

Numerics: convex optimization *à la* Benamou and Brenier

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Primal Problem

Unknowns ($\mathbf{m} = \boldsymbol{\mu}\mathbf{v}$ momentum):

$$\boldsymbol{\mu} : \Omega \times D \to \mathbb{R}_+$$

$$\mathbf{m} : \Omega \times D \to \mathbb{R}^{nd}$$

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Objective

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$$\varphi : \Omega \times D \to \mathbb{R}^n$$

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Dual Problem

Unknown:

$$\varphi : \Omega \times D \to \mathbb{R}^n$$

Objective

$$\max_{\varphi} \left\{ \int_{\partial\Omega} \int_D \varphi(\xi, \cdot) \cdot \mathbf{n}_\Omega(\xi) \boldsymbol{\mu}_b(\xi) \mathrm{d}\xi \right\},$$

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$$\min_{\boldsymbol{\mu},\mathbf{m}} \left\{ \iint_{\Omega \times D} \frac{|\mathbf{m}|^2}{2\boldsymbol{\mu}} \right\},$$

under the constraints

$$\begin{cases} \nabla_\Omega \boldsymbol{\mu} + \nabla_D \cdot \mathbf{m} = 0, \\ \boldsymbol{\mu} = \boldsymbol{\mu}_b \text{ on } \partial\Omega \end{cases}$$

Dual Problem

Unknown:

$$\varphi : \Omega \times D \to \mathbb{R}^n$$

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$$\max_{\varphi} \left\{ \int_{\partial\Omega} \int_D \varphi(\xi, \cdot) \cdot \mathbf{n}_\Omega(\xi) \boldsymbol{\mu}_b(\xi) \mathrm{d}\xi \right\},$$

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$$\nabla_\Omega \cdot \varphi + \frac{1}{2} |\nabla_D \varphi|^2 \leqslant 0.$$

Numerics: convex optimization *à la* Benamou and Brenier

Primal Problem

Unknowns ($\mathbf{m} = \boldsymbol{\mu}\mathbf{v}$ momentum):

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In practice: finite-dimensional “approximation” then **ADMM**.

3. What can be said about these harmonic mappings?

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- The internal energies, e.g.

$$\mu \mapsto \begin{cases} \int_D \mu \ln \mu & \text{if } \mu \text{ has a density w.r.t. Lebesgue,} \\ +\infty & \text{else.} \end{cases}$$

Maximum principle

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Theorem

Take $F : \mathcal{P}(D) \to \mathbb{R} \cup \{+\infty\}$ *convex along generalized geodesics (and few additional regularity property) and a boundary condition* $\boldsymbol{\mu}_b : \partial\Omega \to \mathcal{P}(D)$ *such that* $\sup_{\partial\Omega}(F \circ \boldsymbol{\mu}_b) < +\infty$.

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Then there exists at least one solution $\boldsymbol{\mu}$ *of the Dirichlet problem with boundary conditions* $\boldsymbol{\mu}_b$ *such that*

$$\Delta(F \circ \boldsymbol{\mu}) \geqslant 0 \qquad \text{and} \qquad \operatorname*{ess\,sup}_{\Omega}(F \circ \boldsymbol{\mu}) \leqslant \sup_{\partial\Omega}(F \circ \boldsymbol{\mu}_b).$$

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Already known for harmonic mappings valued in Riemannian manifolds (Ishihara) and Non Positively Curved spaces (Sturm).

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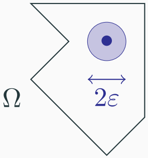

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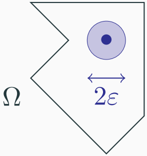

Jensen inequality for Wasserstein barycenters (Agueh, Carlier):

$$F(\boldsymbol{\mu}_\varepsilon(\xi)) \leqslant ⨍_{B(\xi,\varepsilon)} F(\boldsymbol{\mu}_\varepsilon(\eta)) \mathrm{d}\eta.$$

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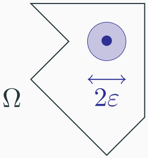

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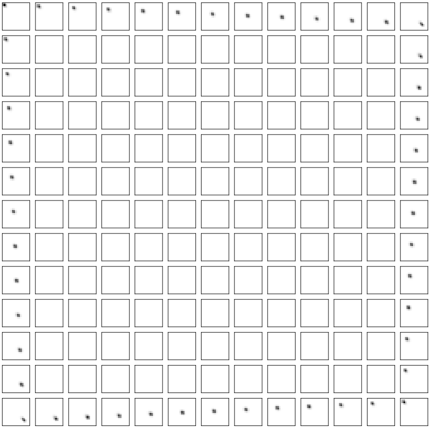
Then limit $\varepsilon \to 0$ to get subharmonicity. □

Case of delta functions

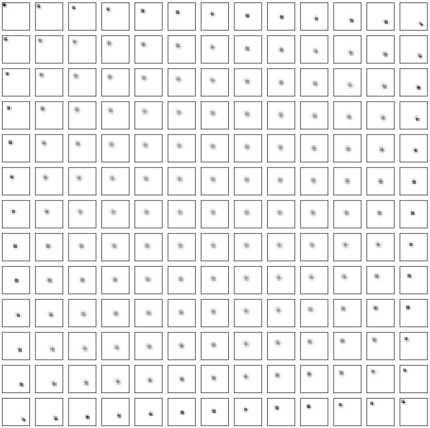
Assume $\boldsymbol{\mu}_b(\xi) = \delta_{f_b(\xi)}$.



Case of delta functions

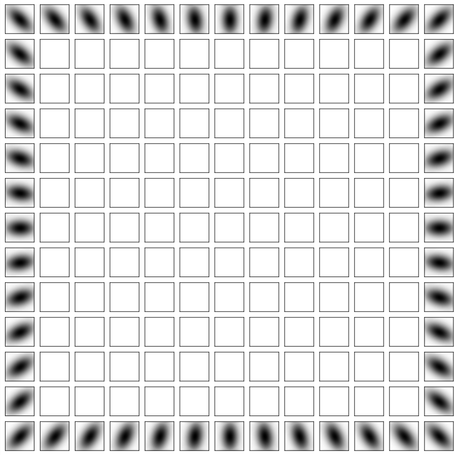
Assume $\boldsymbol{\mu}_b(\xi) = \delta_{f_b(\xi)}$. Then $\boldsymbol{\mu}(\xi) = \delta_{f(\xi)}$ where f is the (usual) harmonic extension of f_b.

Indeed the variance satisfies a maximum principle.



Case of Gaussian measures

Family of “elliptically contoured distributions” $\mathcal{P}_{ec}(D)$, think Gaussians measures.



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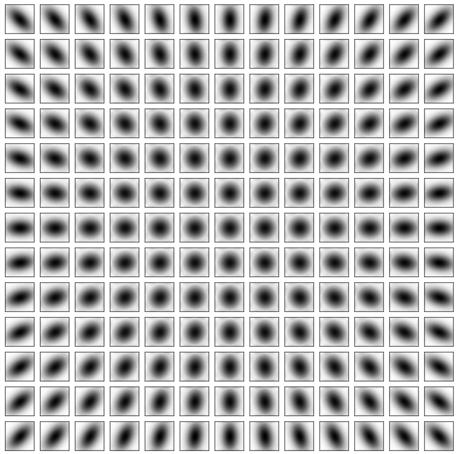

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Let $\boldsymbol{\mu}_b : \partial\Omega \to \mathcal{P}_{ec}(D)$ *Lipschitz such that* $\boldsymbol{\mu}_b(\xi)$ *is not singular for every* $\xi \in \partial\Omega$*.*

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Theorem

Let $\boldsymbol{\mu}_b : \partial\Omega \to \mathcal{P}_{ec}(D)$ Lipschitz such that $\boldsymbol{\mu}_b(\xi)$ is not singular for every $\xi \in \partial\Omega$.

Then there exists a ***unique*** *solution to the Dirichlet problem, it is valued in $\mathcal{P}_{ec}(D)$ and it is* ***smooth****.*

Thank you for your attention