Hidden convexity in a problem of nonlinear elasticity

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1. A non convex problem

2. A convex relaxation

3. Tightness of the relaxation and consequences

1. A non convex problem

A variational problem inspired from elasticity theory

$D \subset \mathbb{R}^d, \ \Omega \subset \mathbb{R}^k$ bounded domain with unit volume and smooth boundary.
$\mathcal{L}_D$ and $\mathcal{L}_\Omega$ Lebesgue measures restricted to D and Ω respectively.

$$\min_{u:D\to\Omega} \left\{ E(u) := \int_D \left(\frac{1}{2}|\nabla u(x)|^2 - f(x)\cdot u(x) \right) \mathrm{d}x \ : \ u = g \text{ on } \partial D \text{ and } u\#\mathcal{L}_D = \mathcal{L}_\Omega \right\}$$

- $f : D \to \mathbb{R}^k$ exterior force.
- $g : \partial D \to \partial\Omega$ prescribed deformation on the boundary.
- $u\#\mathcal{L}_D = \mathcal{L}_\Omega \Leftrightarrow \forall B \subset \Omega, \ \mathcal{L}_D(u^{-1}(B)) = \mathcal{L}_\Omega(B)$.
  If $d = k$ and u smooth and one-to-one, it's equivalent to

$$|\det \nabla u| = 1.$$

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  If $d = k$ and u smooth and one-to-one, it's equivalent to
  $$|\det \nabla u| = 1.$$

Critical points satisfy $\Delta u + f = (\nabla\omega)\circ u$ in the interior of D, where $\omega : \Omega \to \mathbb{R}$ is a Lagrange multiplier.

Without the incompressibilty constraint

$$\min_{u:D\to\mathbb{R}^k} \left\{ \int_D \left(\frac{1}{2}|\nabla u(x)|^2 - f(x)\cdot u(x) \right) \mathrm{d}x \ : \ u = g \text{ on } \partial D \text{ and } \sout{u\#\mathcal{L}_D = \mathcal{L}_\Omega} \right\}$$

It's a convex problem.

Theorem

Under mild regularity assumptions on f, g and D, there exists a unique global minimizer u and it satisfies

$$\begin{cases} \Delta u + f = 0 & \text{in } D, \\ u = g & \text{on } \partial D. \end{cases}$$

Without Dirichlet energy and $D = \Omega$ convex

$$\min_{u:D\to D} \left\{ \int_D \left(\frac{1}{2}|\nabla u(x)|^2 - f(x) \cdot u(x) \right) \mathrm{d}x \ : \ u = g \text{ on } \partial D \text{ and } u\#\mathcal{L}_D = \mathcal{L}_D \right\}$$

Theorem (polar factorization)[1]

We assume $f \in L^2(D, \mathbb{R}^k)$ and $(f\#\mathcal{L}_D)(B) = 0$ if $\mathcal{H}^{d-1}(B) = 0$. Then f can be uniquely written

$$f = (\nabla \omega) \circ u$$

where $\omega : D \to \mathbb{R}$ convex and $u : D \to D$ satisfies $u\#\mathcal{L}_D = \mathcal{L}_D$. The function u is the unique minimizer of the energy under the constraint $u\#\mathcal{L}_D = \mathcal{L}_D$.

The mapping $\nabla\omega$ is the optimal transport map of $\mathcal{L}_D$ onto $f\#\mathcal{L}_D$ for the quadratic cost.

[1] Brenier (1987). *Décomposition polaire et réarrangement monotone des champs de vecteurs.*

Goal of this talk

An existence theory exists for instance in the framework of polyconvex functionals. [2].

[2] Ball (1976). *Convexity conditions and existence theorems in nonlinear elasticity.*

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Today: propose a convex relaxation of the problem with Dirichlet energy **and** incompressibility constraint.

Previous attempts:

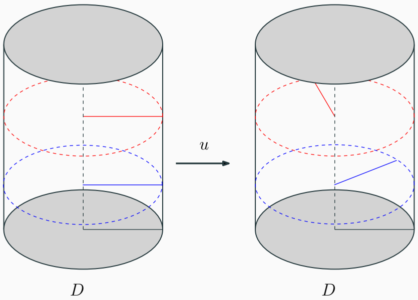
- J. Louet (2014). *Optimal transport problems with gradient penalization.*
- R. Awi and W. Gangbo (2014). *A polyconvex integrand; Euler–Lagrange equations and uniqueness of equilibrium.*
- T. Mollenhoff and D. Cremers (2019). *Lifting vectorial variational problems: a natural formulation based on geometric measure theory and discrete exterior calculus.*

Very recent paper on uniqueness and regularity:

- W. Gangbo, M. Jacobs and I. Kim (2020). *Well-posedness and regularity for a polyconvex energy.*

[2] Ball (1976). *Convexity conditions and existence theorems in nonlinear elasticity.*

An example: pure torsion of a cylinder



$D = \Omega = B(0,1) \times [0,1]$. For $a > 0$,

$$u_a \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} R_{az} \begin{pmatrix} x \\ y \end{pmatrix} \\ z \end{pmatrix}$$

where $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ rotation by an angle θ.

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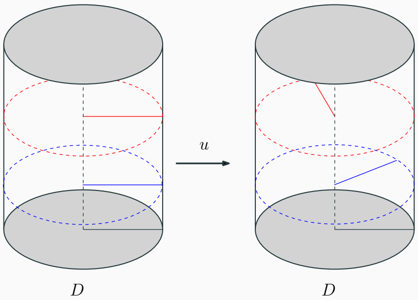

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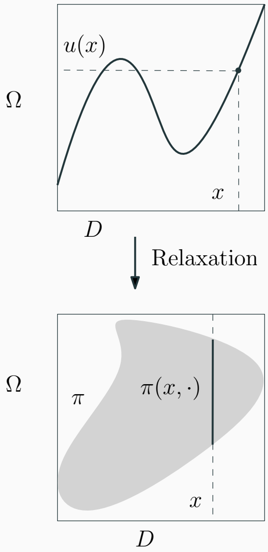
Result $(f \equiv 0)$

For all $a > 0$, the function u_a is a critical point of the energy.

We will see that, at least for small a, it is a global minimizer with boundary condition $g = u_a|_{\partial D}$.

2. A convex relaxation

Transport plan



Method

$u : D \to \Omega$ satisfying $u\#\mathcal{L}_D = \mathcal{L}_\Omega$ is replaced by $\pi \in \mathcal{P}(D \times \Omega)$ whose marginals are $\mathcal{L}_D$ and $\mathcal{L}_\Omega$. We write $\pi \in \Pi(\mathcal{L}_D, \mathcal{L}_\Omega)$.

- The marginal constraints are linear. For instance, for all $a \in C(D)$:

$$\iint_{D\times\Omega} a(x)\,\pi(\mathrm{d}x, \mathrm{d}y) = \int_D a(x)\,\mathrm{d}x$$

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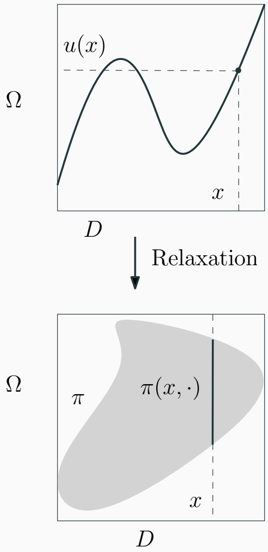

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- By disintegration/fubinization, one can see $\pi \in \Pi(\mathcal{L}_D, \mathcal{L}_\Omega)$ as a mapping $\pi : x \in D \to \pi(x, \cdot) \in \mathcal{P}(\Omega)$.
- If $u : D \to \Omega$, we can define π_u by $\pi_u(x, \cdot) = \delta_{y=u(x)}$.

Back to the problem

$$\min_{u:D\to\Omega} \left\{ \int_D \left(\frac{1}{2}|\nabla u(x)|^2 - f(x)\cdot u(x) \right) \mathrm{d}x \;:\; u = g \text{ on } \partial D \text{ and } u\#\mathcal{L}_D = \mathcal{L}_\Omega \right\}$$

$$\downarrow$$

$$\min_{\pi\in\mathcal{P}(D\times\Omega)} \left\{ ?? - \iint_{D\times\Omega} (f(x)\cdot y)\ \pi(\mathrm{d}x,\mathrm{d}y) \;:\; ?? \text{ and } \pi\in\Pi(\mathcal{L}_D,\mathcal{L}_\Omega) \right\}$$

Without Dirichlet energy, it's exactly the relaxation used by Yann Brenier in 1987 to prove polar factorization!

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Question

How to define a Dirichlet energy for $\pi : D \to \mathcal{P}(\Omega)$?

Dirichlet energy for measure-valued mappings[3] [4]

$J : D \times \Omega \to \mathbb{R}^{d \times k}$ matrix-valued measure $\frac{\mathrm{d}J}{\mathrm{d}\pi}$ "density of Jacobian matrix".
Constraint between π and J:

$$\nabla_x \pi(x, y) + \nabla_y \cdot J(x, y) = 0,$$

and the Dirichlet energy is

$$\iint_{D \times \Omega} \frac{|J|^2}{2\pi} = \iint_{D \times \Omega} \frac{1}{2} \left| \frac{\mathrm{d}J}{\mathrm{d}\pi} \right|^2 \mathrm{d}\pi$$

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[4] Lavenant (2019). *Harmonic mappings valued in the Wasserstein space*.

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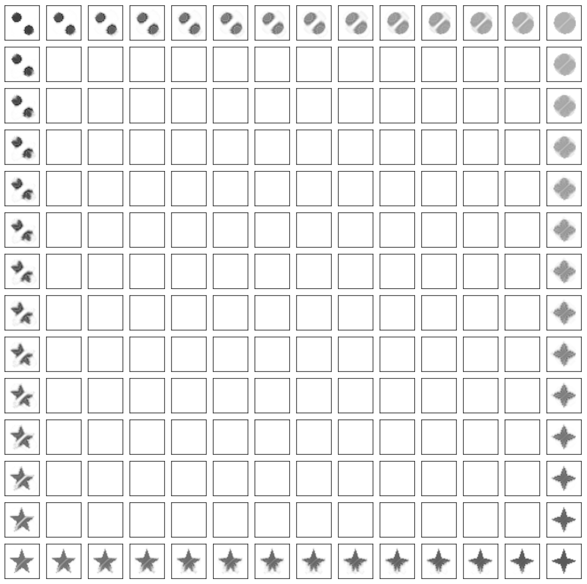
If $\pi(\mathrm{d}x,\mathrm{d}y) = \pi_u(\mathrm{d}x,\mathrm{d}y) = \delta_{y=u(x)}\mathrm{d}x$, we choose

$$\frac{\mathrm{d}J_u}{\mathrm{d}\pi_u}(x,y) = \nabla u(x) \quad \text{so that} \quad \iint_{D\times\Omega} \frac{|J_u|^2}{2\pi_u} = \int_D \frac{1}{2}|\nabla u(x)|^2 \mathrm{d}x.$$

This definition has a link with a Dirichlet energy "à la Korevaar and Schoen" for mappings valued in metric spaces.

[3] Brenier (2003). *Extended Monge-Kantorovich theory.*
[4] Lavenant (2019). *Harmonic mappings valued in the Wasserstein space.*

Harmonic mappings without the incompressibility constraint



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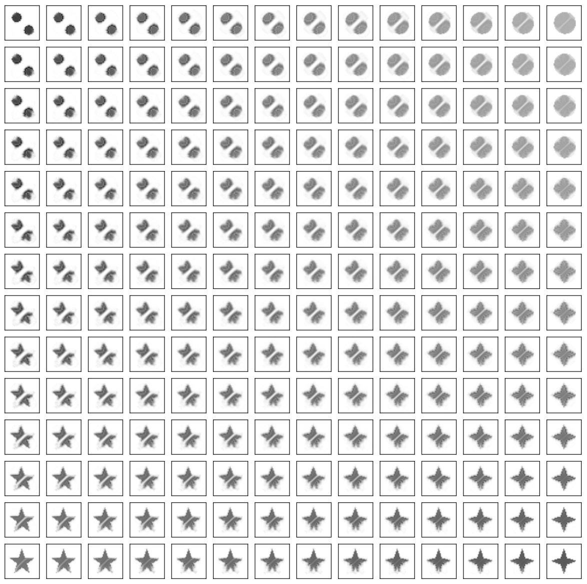

Back to the problem with incompressibility: the relaxation

Relaxed (primal) problem

We say that $\pi \in \mathcal{P}(D \times \Omega)$ and $J \in \mathcal{M}(D \times \Omega, \mathbb{R}^{d \times k})$ are admissible if

$$\begin{cases} \pi & \in & \Pi(\mathcal{L}_D, \mathcal{L}_\Omega) & \\ \nabla_x \pi + \nabla_y \cdot J & = & 0 & \text{in } D \times \Omega \\ \pi(x, \cdot) & = & \delta_{y=g(x)} & \text{for } x \in \partial D \end{cases}$$

and we define

$$E_r(\pi, J) = \iint_{D \times \Omega} \frac{1}{2} \left(\left| \frac{\mathrm{d}J}{\mathrm{d}\pi} \right|^2 (x, y) - f(x) \cdot y \right) \pi(\mathrm{d}x, \mathrm{d}y)$$

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Minimizing E_r among the admissible (π, J) is a convex problem!

If $u \# \mathcal{L}_D = \mathcal{L}_\Omega$ and $u = g$ on ∂D then (π_u, J_u) admissible and

$$E(u) = E_r(\pi_u, J_u).$$

The dual problem

Three Lagrange multipliers: ψ, ω for the marginal constraints, φ for $\nabla_x \pi + \nabla_y \cdot J = 0$ and the boundary condition.

The dual problem

We say that (φ, ψ, ω), where $\varphi \in C^1(D \times \Omega, \mathbb{R}^d)$, $\psi \in C(D)$ and $\omega \in C(\Omega)$, is admissible if for all $(x, y) \in D \times \Omega$,

$$\psi(x) + \omega(y) - f(x) \cdot y \geqslant \left(\nabla_x \cdot \varphi + \frac{1}{2} |\nabla_y \varphi|^2 \right) (x, y).$$

Then we define

$$E_r^*(\varphi, \psi, \omega) = \int_{\partial D} \varphi(x, g(x)) \cdot \mathbf{n}_D(x) \, \mathrm{d}x - \int_D \psi(x) \, \mathrm{d}x - \int_\Omega \omega(y) \, \mathrm{d}y.$$

Link between the primal and the dual

Proposition (weak duality)

If (π, J) admissible in the primal and (φ, ψ, ω) admissible in the dual,

$$E_r^*(\varphi, \psi, \omega) \leqslant E_r(\pi, J).$$

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Moreover, equality holds if and only if

$$\begin{cases} \nabla_y \varphi(x,y) = \dfrac{\mathrm{d}J}{\mathrm{d}\pi}(x,y) & \text{for } \pi - \text{a.e. } x, y \\ \psi(x) + \omega(y) - f(x) \cdot y = \nabla_x \cdot \varphi + \dfrac{1}{2}|\nabla_y \varphi|^2 & \text{for } \pi - \text{a.e. } x, y. \end{cases}$$

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Strong duality: not proven, but should be doable with Fenchel Rockaffellar theorem.

Existence of an optimal dual solution is an open problem (already in the case without incompressibility).

A more general case

Now $W : \mathbb{R}^{d\times k} \to \mathbb{R}$ convex density of elastic energy and $\Phi : \mathcal{P}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ convex function of measures.

$$\min_{u:D\to\Omega} \left\{ \int_D (W(\nabla u(x)) - f(x) \cdot u(x))\, \mathrm{d}x + \Phi(u\#\mathcal{L}_D) \ : \ u = g \text{ on } \partial D \right\}.$$

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Previous case $W(C) = \frac{1}{2}|C|^2$ and $\Phi(\mu) = \begin{cases} 0 & \text{if } \mu = \mathcal{L}_\Omega \\ +\infty & \text{otherwise} \end{cases}$

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Compressible case: take $h : [0, +\infty) \to \mathbb{R} \cup \{+\infty\}$ convex, then

$$\int_D h(\det \nabla u(x))\mathrm{d}x = \Phi(u\#\mathcal{L}_D),$$

for some Φ convex on the set of measures.

Relaxation in the general case

$\mu = u\#\mathcal{L}_D$ becomes a variable to optimize.

Primal

(π, J, μ) admissible if

$$\begin{cases} \pi \in \Pi(\mathcal{L}_D, \mu) \\ \nabla_x \pi + \nabla_y \cdot J = 0 \text{ in } D \times \Omega \\ \pi(x, \cdot) = \delta_{y=g(x)} \text{ on } \partial D \end{cases}$$

and

$$E_r(\pi, J, \mu) = \iint \frac{1}{2}\left(W\left(\frac{\mathrm{d}J}{\mathrm{d}\pi}\right) - f \cdot y\right)\pi + \Phi(\mu).$$

Dual

(φ, ψ, ω) admissible if

$$\psi(x) + \omega(y) - f(x) \cdot y \geqslant \nabla_x \cdot \varphi + W^*\left(\nabla_y \varphi\right).$$

and

$$E_r^*(\varphi, \psi, \omega) = \int_{\partial D} \varphi(g) \cdot \mathbf{n}_D - \int_D \psi - \Phi^*(\omega).$$

Complementary slackness impose $\omega \in \partial\Phi(\mu)$ at optimality.

3. Tightness of the relaxation and consequences

Strategy

Combining the embedding $u \mapsto (\pi_u, J_u, u\#\mathcal{L}_D)$ and weak duality,

$$\min_{u:D\to\Omega} E(u) \geqslant \min_{(\pi,J,\mu)\text{ admissible}} E_r(\pi, J, \mu) \geqslant \sup_{(\varphi,\psi,\omega)\text{ admissible}} E_r^*(\varphi, \psi, \omega).$$

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Important remark

If for $u : D \to \Omega$ we can find (φ, ψ, ω) admissible such that

$$E_r^*(\varphi, \psi, \omega) = E(u)$$

then u is a global minimizer of the energy E and $(\pi_u, J_u, u\#\mathcal{L}_D)$ minimizes the relaxed energy.

In this case, the relaxation doesn't give better competitors.

Convexity of the pressure

Theorem

Let $u : D \to \Omega$ a smooth function satisfying $u = g$ on ∂D and $\omega \in \partial\Phi(u\#\mathcal{L}_D)$ such that

$$\nabla \cdot (DW(\nabla u)) + f = (\nabla\omega) \circ u.$$

If ω can be extended on $\mathbb{R}^k$ in a (strictly) **convex** function then u is a (the unique) global minimizer global of the energy and $(\pi_u, J_u, u\#\mathcal{L}_D)$ minimizes the relaxed energy.

$\partial\Phi(\mu)$ is the subdifferential of the functional Φ at the point μ. If Φ is the incompressibility constraint, $\partial\Phi(\mathcal{L}_\Omega) = C(\Omega)$, and $\partial\Phi(\mathcal{L}_\Omega) = \emptyset$ for $\mu \neq \mathcal{L}_\Omega$.

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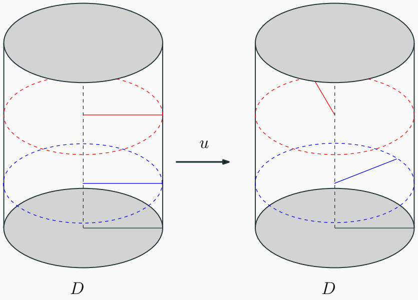
Idea of the proof. Same competitor then Brenier (in the case $\Phi \equiv 0$) with $\varphi(x, y) = y^\top \nabla u(x)$, and ω given by the optimality conditions of u.

Optimal assumption?



Torsion of the cylinder : $2\pi/a$ vertical period.

If W Dirichlet energy, the pressure is

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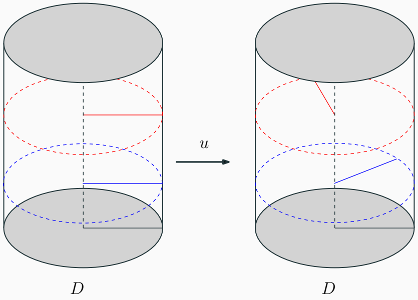

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In general, in the case $f = 0$, the linearization of the problem yields Stokes equations:

$$\begin{cases} \Delta u & = \nabla p \\ \nabla \cdot u & = 0 \end{cases}$$

and $p \sim \omega$ satisfies $\Delta p = 0$, it is not convex generically.

An improvement

We restrict to the case Dirichlet case $W(C) = {}^1\!/_2|C|^2$. Let $\lambda_1(D) > 0$ be the first eigenvalue of the Dirichlet Laplacian on D.

Theorem

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If ω can be extended on $\mathbb{R}^k$ in a **λ-convex** function **with** $\lambda > -\lambda_1(D)$ then u is the unique global minimizer global of the energy and $(\pi_u, J_u, u\#\mathcal{L}_D)$ minimizes the relaxed energy.

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Idea of the proof. There was still some leeway in Brenier's competitor.

A simpler proof of global optimality

It is enough to replace Φ by its linear approximation in the space of measures:

$$E(u) = \int_D (W(\nabla u) - f \cdot u) + \Phi(u\#\mathcal{L}_D)$$

A simpler proof of global optimality

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$$E(u) = \int_D (W(\nabla u) - f \cdot u) + \Phi(u\#\mathcal{L}_D)$$

$$\underbrace{\geqslant}_{= \text{ if } \omega \in \partial\Phi(u\#\mathcal{L}_D)} \int_D (W(\nabla u) - f \cdot u) + \int_\Omega \omega \, \mathrm{d}(u\#\mathcal{L}_D) - \Phi^*(\omega)$$

$$= \int_D (W(\nabla u) - f \cdot u) + \int_D \omega \circ u - \Phi^*(\omega) =: \tilde{E}_\omega(u) - \Phi^*(\omega).$$

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If u is a global minimizer of $\tilde{E}_\omega$ for $\omega\in\partial\Phi(u\#\mathcal{L}_D)$ then it is a global minimizer of E. The energy $\tilde{E}_\omega$ is convex under convexity assumption on ω.

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The improvement from ω convex to ω λ-convex works if W is uniformly convex.

Conclusion

Results :

- Convex relaxation of a non-convex problem.
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Thank you for your attention

Link with a metric definition *à la* Korevaar and Schoen[5]

Take a convex Ω. Quadratic Wasserstein distance on $\mathcal{P}(\Omega)$:

$$\mathcal{W}(\mu,\nu) = \min_{\gamma \in \mathcal{P}(\Omega\times\Omega)} \left\{ \iint |y-z|^2 \, \gamma(\mathrm{d}y, \mathrm{d}z) \ : \ \gamma \in \Pi(\mu,\nu) \right\}.$$

Theorem

Let $\pi \in \mathcal{P}(D \times \Omega)$ whose first marginal is $\mathcal{L}_D$. Then

$$\min_J \left\{ \iint_{D\times\Omega} \frac{|J|^2}{2\pi} \ : \ \nabla_x \pi + \nabla_y \cdot J = 0 \right\} = \lim_{\varepsilon\to 0} C_d \iint_{D\times D} \frac{\mathcal{W}^2(\pi(x,\cdot), \pi(x',\cdot))}{2\varepsilon^{d+2}} \mathbb{1}_{|x-x'|\leqslant\varepsilon} \, \mathrm{d}x\mathrm{d}x'$$

where C_d constant depending only on d.

[5] Korevaar and Schoen (1993). *Sobolev spaces and harmonic maps for metric space targets.*