Variational Mean Field Games: on estimates on the density and the pressure and their consequences for the Lagrangian point of view

Hugo Lavenant[a]

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[a]Department of Mathematics, University of British Columbia

Variational Mean Field Games of first order [1]

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where $\rho : [0,T] \to \mathcal{P}(\Omega)$ and $v : [0,T] \times \Omega \to \mathbb{R}^d$ while

$$\partial_t \rho + \nabla \cdot (\rho v) = 0.$$

The initial density ρ_0 is given, $V, \Psi : \Omega \to \mathbb{R}$ are potentials.

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The function $F:\mathcal{P}(\Omega)\to\mathbb{R}$ is convex. Two cases:

$$F(\rho)=\begin{cases}\int_\Omega f(\rho)\mathrm{d}x & \text{"soft congestion"},\\ 0 \text{ if } \rho\leqslant 1,\ +\infty \text{ otherwise} & \text{"hard congestion"}.\end{cases}$$

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Mean Field Games: Lagrangian interpretation

If ρ is a minimizer of the problem, there exists $Q \in \mathcal{P}(C([0,T],\Omega))$ such that $Q(\gamma)\mathrm{d}\gamma$ represents the proportion of agents following the strategy γ.

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- for all t, $e_t \# Q = \rho_t$,
- Q-a.e. curve γ solves the control problem

$$\min_{\omega \text{ s.t. } \omega(0)=\gamma(0)} \left[\int_0^T \left(\frac{1}{2}|\dot{\omega}_t|^2 + V(\omega_t) + p_t(\omega_t) \right) \mathrm{d}t + \Psi(\omega_T) \right].$$

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The field $p : [0,T] \times \Omega \to \mathbb{R}$ is the price or the pressure.

$$p_t(x) \quad \begin{cases} = f'(\rho_t(x)) & \text{(Soft congestion)} \\ \geqslant 0 \text{ and } = 0 \text{ if } \rho_t(x) < 1 & \text{(Hard congestion)} \end{cases}$$

Mean Field Games

Mean Field Games of first order with local couplings:

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- Each agent tries to minimize Ψ at the final time but try to avoid the others.
- The effect of the other agents is only felt by their mean field effect through the price p.
- Quadratic (because on the square on the velocity), first order (no noise) and potential (comes from the minimization of a functional).

Mean Field Games: Eulerian interpretation

Value function for a single agent:

$$\varphi(t,x) := \min_{\omega \text{ s.t. } \omega(t)=x} \left[\int_t^T \left(\frac{1}{2}|\dot{\omega}_t|^2 + V(\omega_t) + p_t(\omega_t) \right) \mathrm{d}t + \Psi(\omega_T) \right].$$

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It solves a Hamilton-Jacobi equation. The optimal control is $\dot{\gamma}(t) = -\nabla\varphi(t,\gamma(t)) = v(t,\gamma(t))$.

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho v) & = 0, \\ \rho_0 & \text{given}, \\ -\partial_t \varphi + \frac{1}{2}|\nabla\varphi|^2 & = V + p, \\ \varphi(T,\cdot) & = \Psi, \\ v & = -\nabla\varphi. \end{cases}$$

Regularity issues

Abstract idea of Mean Field Games

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Variational problem
Easy existence of ρ, v, p, φ

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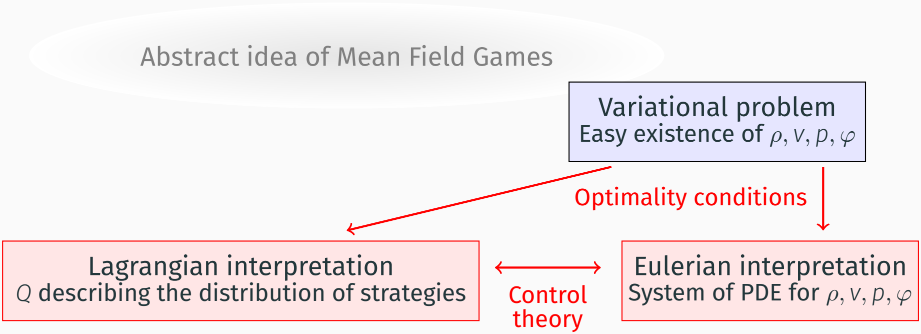
Abstract idea of Mean Field Games

Variational problem
Easy existence of ρ, v, p, φ

Lagrangian interpretation
Q describing the distribution of strategies

Eulerian interpretation
System of PDE for ρ, v, p, φ

Regularity issues



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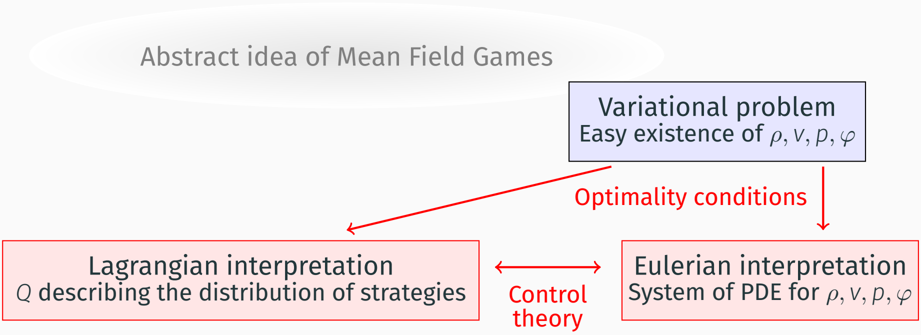

Questions

What regularity can be deduced on ρ on p from the variational formulation?

Does it justify the Lagrangian and Eulerian systems?

Regularity of ρ: soft congestion [2]

$$\min_{\rho, v} \left[\frac{1}{2} \int_0^T \int_\Omega \frac{1}{2} |v_t|^2 \rho_t \, \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega V \rho_t \, \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega f(\rho_t) \, \mathrm{d}x\mathrm{d}t + \int_\Omega \Psi \rho_T \, \mathrm{d}x. \right]$$

with $\partial_t \rho + \nabla \cdot (\rho v) = 0$.

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Theorem

Assume V is Lipschitz, $\Psi \in C^{1,1}$ and $f''(s) \geqslant s^\alpha$ with $\alpha \geqslant -1$. Then, for every $t < T$, the measure ρ belongs to $L^\infty([t, T] \times \Omega)$.

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Corollary

Under the assumption of the previous theorem, if f' is bounded from below then $p=f'(\rho)$ belongs to $L^\infty([t,T]\times\Omega)$.

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$$\sim C(m)\int_\Omega \left|\nabla\left(\rho^{(m+1+\alpha)/2}\right)\right|^2$$

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If $m > 1$, with $\beta > 1$ such that $H^1(\Omega) \hookrightarrow L^{2\beta}(\Omega)$,

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Integration with respect to time and *Moser iterations*. □

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d is the dimension of the ambient space.

Theorem

Assume $\nabla V\in L^q$ with $q>d$. Then p belongs to $L^\infty([0,T)\times\Omega)$ with a norm depending only on $\|\nabla V\|_{L^q}$ and Ω.

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The proof relies on an inequality

$$\Delta(p + V) \geqslant -D_{tt} \ln \rho \underbrace{\geqslant}_{\text{on } \{p>0\}} 0.$$

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Consequences: regularity of the value function [4] [5]

Previous results on the regularity of φ solving:

$$-\partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2 = g, \quad g \in L^\infty$$

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- φ is Hölder-continuous on $(0, T) \times \Omega$,
- $\partial_t \varphi \in L^{1+\varepsilon}$ and $\nabla \varphi \in L^{2+\varepsilon}$,
- The Hamilton-Jacobi equation is satisfied in the almost everywhere sense.

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Consequence: Lagrangian point of view [6] [7]

Case of soft congestion with a pressure in $L^\infty((0,T] \times \Omega)$.

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Link between Eulerian and Lagrangian point of view

The Eulerian value function φ is in fact a Lagrangian value function. For $t > 0$, for ρ_t a.e. x,

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Thank you for your attention