Dynamical Optimal Transport: discretization and convergence

Hugo Lavenant[a]

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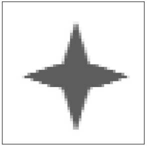
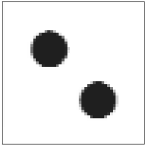
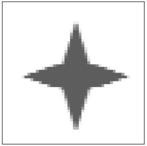
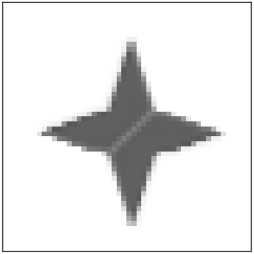
PIMS–AMI seminar. University of Alberta, Edmonton

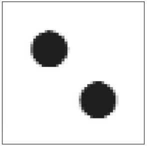
[a]Department of Mathematics, University of British Columbia

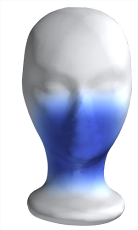


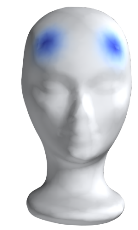


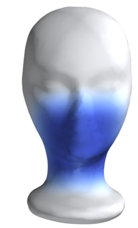
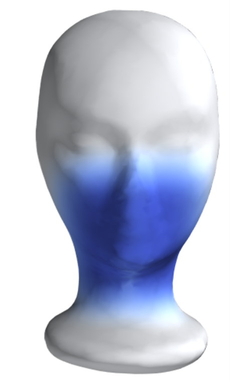
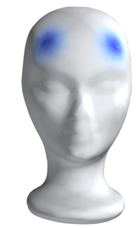










1. Dynamical Optimal transport

2. Discretization on discrete surfaces (with S. Claici, E. Chien and J. Solomon)[1]

3. A general framework for convergence[2]

[1] H. Lavenant, S. Claici, E. Chien and J. Solomon, *Dynamical Optimal Transport on Discrete Surfaces*. `Arxiv 1809.07083`.

[2] H. Lavenant, *Unconditional convergence for discretizations of dynamical optimal transport*. `Arxiv 1909.08790`.

1. Dynamical Optimal transport

Static formulation of optimal transport

(X, g) compact Riemannian manifold possibly with boundary, the geodesic distance is d_g.

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Definition

Let $\mu, \nu \in \mathcal{P}(X)$ be two probability measures on X. The **static** optimal transport problem is

$$\min_{\pi} \iint_{X \times X} d_g(x, y)^2 \, \pi(\mathrm{d}x, \mathrm{d}y),$$

where the minimum is taken over all probability measures on $X \times X$ whose marginals are μ and ν.

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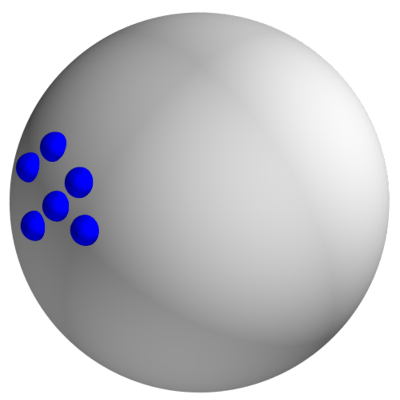
The minimal value is $W_2^2(\mu, \nu)$ the squared **Wasserstein distance** between μ and ν, which metrizes weak convergence on $\mathcal{P}(X)$.

From static to dynamic



$$\mu = \sum_i a_i \delta_{x_i},$$

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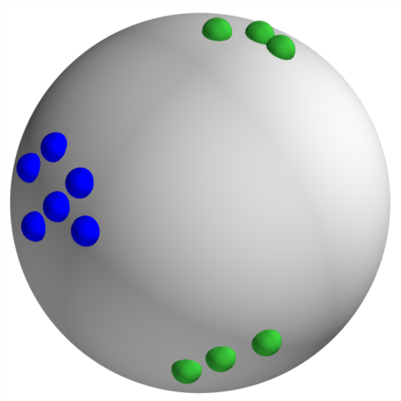

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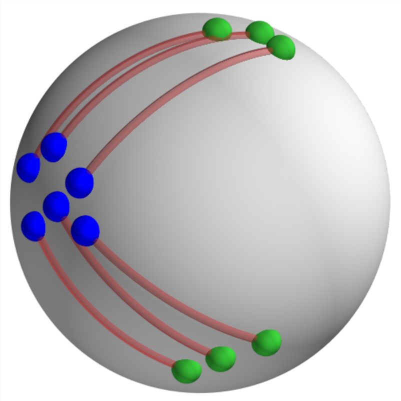

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Solve the **Linear Programming problem**

$$\min_{\pi} \sum_{i,j} \pi_{ij} \, d_g(x_i, y_j)^2$$

with conservation of mass constraints

$$\begin{cases} \sum_j \pi_{ij} = a_i, \\ \sum_i \pi_{ij} = b_j, \end{cases}$$

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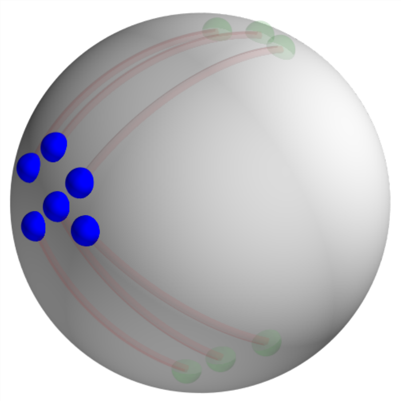

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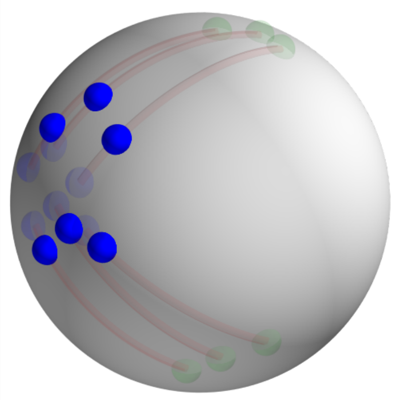

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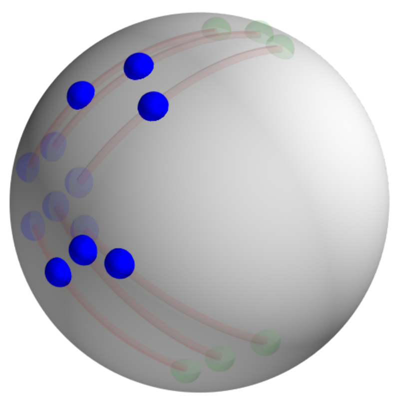

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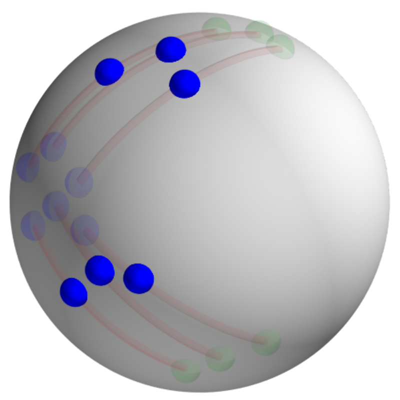

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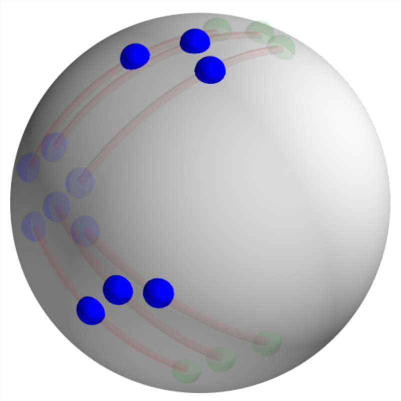

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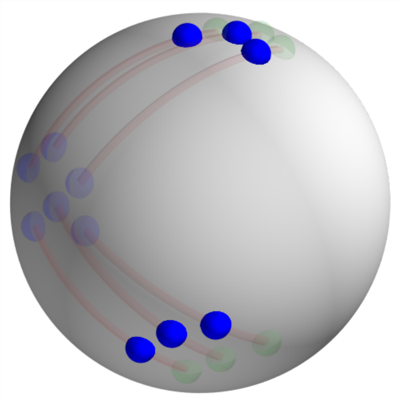

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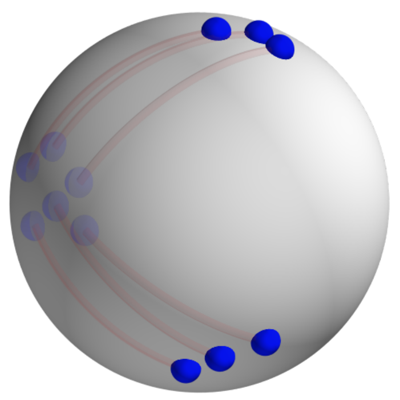

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$$\min_{\rho, \mathbf{v}} \int_0^1 \int_X |\mathbf{v}(t,x)|^2 \rho(t,x) \, \mathrm{d}t \, \mathrm{d}x$$

where the minimum is taken over densities $\rho : [0,1] \times X \to \mathbb{R}_+$ and velocity fields $\mathbf{v} : [0,1] \times X \to TX$ such that

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \\ \rho(0, \cdot) = \mu, \quad \rho(1, \cdot) = \nu. \end{cases}$$

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The two problems are equivalent: the values are the same and one can construct minimizers from one formulation by the knowledge of minimizers of the other (Benamou and Brenier, 2000).

Convex formulation

Change of variables $\mathbf{m} = \rho \mathbf{v}$ the **momentum** the unknown.

Proper framework $\rho \in \mathcal{M}_+([0,1] \times X)$ and $\mathbf{m} \in \mathcal{M}([0,1] \times X, TX)$.

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$$= \sup_{a,\mathbf{b} \text{ continuous}} \left\{ \langle a, \rho \rangle + \langle \mathbf{b}, \mathbf{m} \rangle \ : \ a + \frac{1}{2}|\mathbf{b}|^2 \leqslant 0 \text{ on } [0,1] \times X \right\}.$$

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The continuity equation becomes linear and is understood in a weak sense.

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Remark

Existence comes from the direct method of calculus of variations. Uniqueness holds if μ or ν is absolutely continuous with respect to the volume measure.

About regularity

Take $\mu, \nu \in \mathcal{P}(X)$ and $(\rho, \mathbf{m})$ solution of the optimal transport problem.

Theorem (Smoothness: Caffarelli and others (1990 and later))

Assume X is the torus or a bounded domain of a Euclidean space with convex boundary.

If μ, ν are smooth and bounded from below by a strictly positive constant, then ρ and $\mathbf{m}$ are smooth.

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On a generic Riemannian manifold, smoothness of the data does not imply smoothness of the interpolation (Ma–Trudinger–Wang, Loeper, Kim, etc.).

On the absence of lower bounds

Take $\mu, \nu \in \mathcal{P}(X)$ and $(\rho, \mathbf{m})$ solution of the optimal transport problem.

Counterexample (Santambrogio and Wang (2016))

Let X be a convex domain of the Euclidean space with smooth boundary. There exists μ, ν smooth and bounded from below by a strictly positive constant such that

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Counterexample

Let X be the 2-dimensional torus. For every $\varepsilon > 0$, there exists μ, ν smooth and bounded from below by a strictly positive constant such that

$$\min_{[0,1]\times X} \rho \leqslant \varepsilon \left(\min_X \mu, \min_X \nu \right).$$

2. Discretization on discrete surfaces (with S. Claici, E. Chien and J. Solomon)[a]

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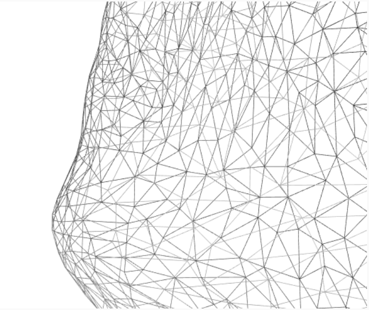
Discrete surfaces



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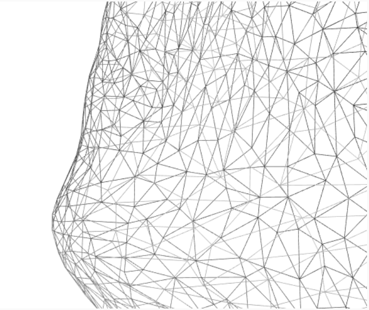

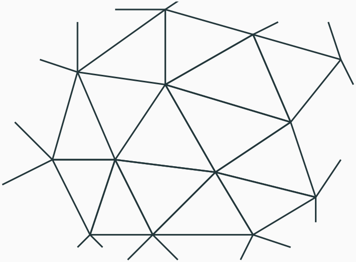

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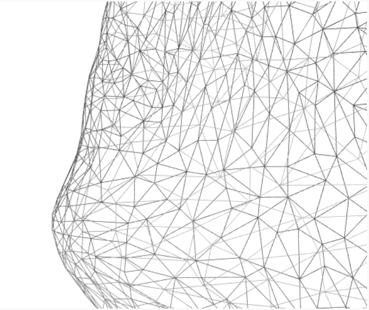

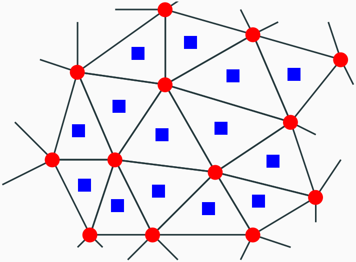

$\rho : \bullet, \ \mathbf{m} : \blacksquare.$

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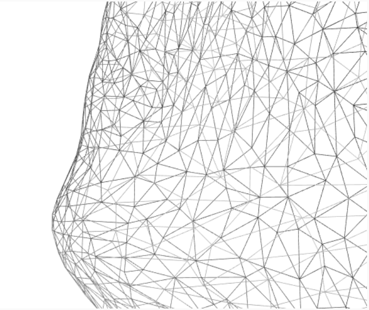

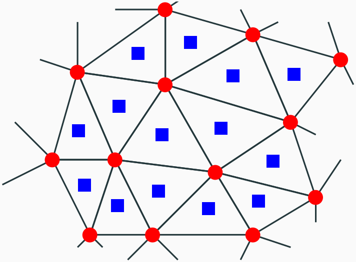

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Continuity equation: $\rho \in \mathbb{P}^1_{\text{time}}\mathbb{P}^1_{\text{space}}$ and $\mathbf{m} \in \mathbb{P}^0_{\text{time}}\mathbb{P}^0_{\text{space}}$.

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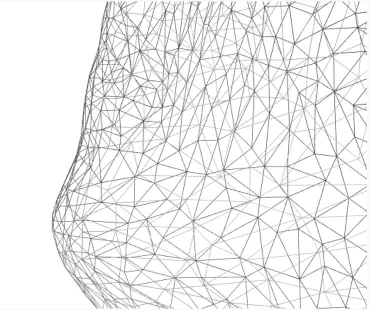

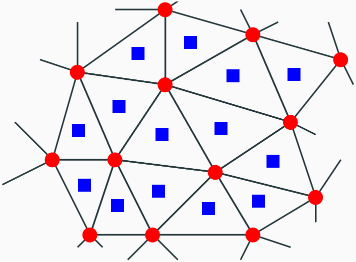

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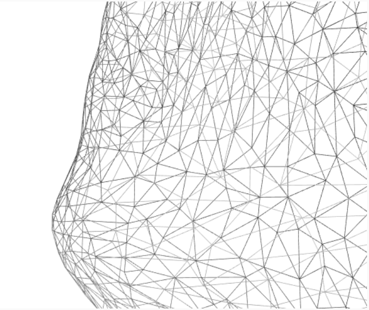

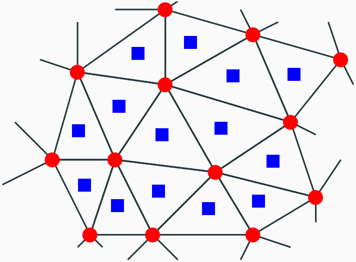

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Objective functional: if $\mathcal{G}$ is the space-time grid over which $\mathbf{m}$ is defined,

$$\iint_{[0,1]\times X} \frac{|\mathbf{m}|^2}{2\rho} \sim \frac{1}{2} \sum_{(t,x)\in\mathcal{G}} \frac{|\mathbf{m}_{t,x}|^2}{} \mathrm{vol}((t,x)).$$

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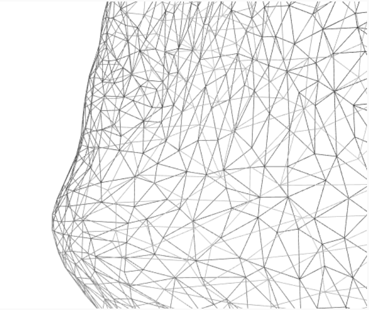

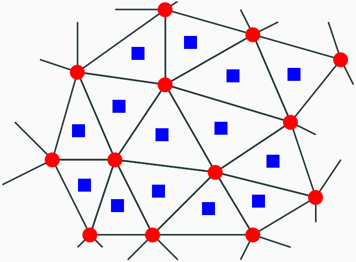

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$$\iint_{[0,1]\times X} \frac{|\mathbf{m}|^2}{2\rho} \sim \frac{1}{2} \sum_{(t,x)\in\mathcal{G}} \frac{|\mathbf{m}_{t,x}|^2}{[\text{Average of } \rho \text{ around } (t,x)]} \mathrm{vol}((t,x)).$$

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Alternatives: proximal splitting (Papadakis *et al.*, 2014), Helmholtz-Hodge decomposition (Henry *et al.*, 2019).

Examples

Positivity and **mass preservation** are automatically enforced

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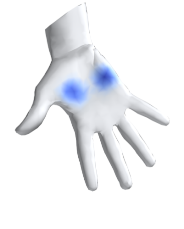
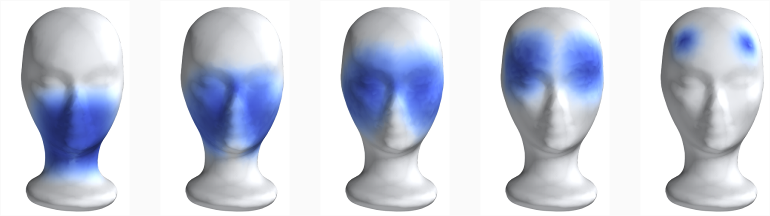


Not so perfect?



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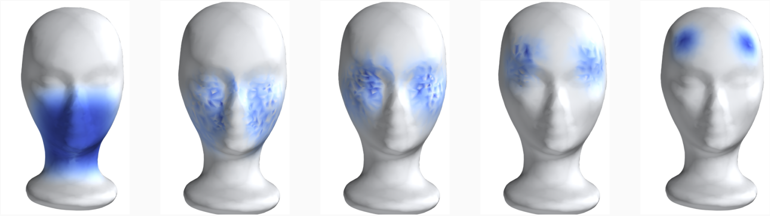

3. A general framework for convergence[a]

[a] H. Lavenant, *Unconditional convergence for discretizations of dynamical optimal transport*. `Arxiv 1909.08790`.

A generic discretization

Original problem

Unknowns:

$$\rho : [0,1] \times X \to \mathbb{R}_+$$

$$\mathbf{m} : [0,1] \times X \to TX$$

Objective

$$\min_{\rho,\mathbf{m}} \left\{ \iint_{[0,1]\times X} \frac{|\mathbf{m}|^2}{2\rho} \right\}$$

under the constraints

$$\begin{cases} \partial_t \rho + \nabla \cdot \mathbf{m} = 0, \\ \rho(0,\cdot) = \mu,\ \rho(1,\cdot) = \nu. \end{cases}$$

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$\mathcal{X}_\sigma, \mathcal{Y}_\sigma$ vector spaces (stand for $\mathcal{M}(X), \mathcal{M}(TX)$),
$\mathrm{Div}_\sigma : \mathcal{Y}_\sigma \to \mathcal{X}_\sigma$ linear operator,
$A_\sigma : \mathcal{X}_\sigma \times \mathcal{Y}_\sigma \to [0, +\infty]$ convex,
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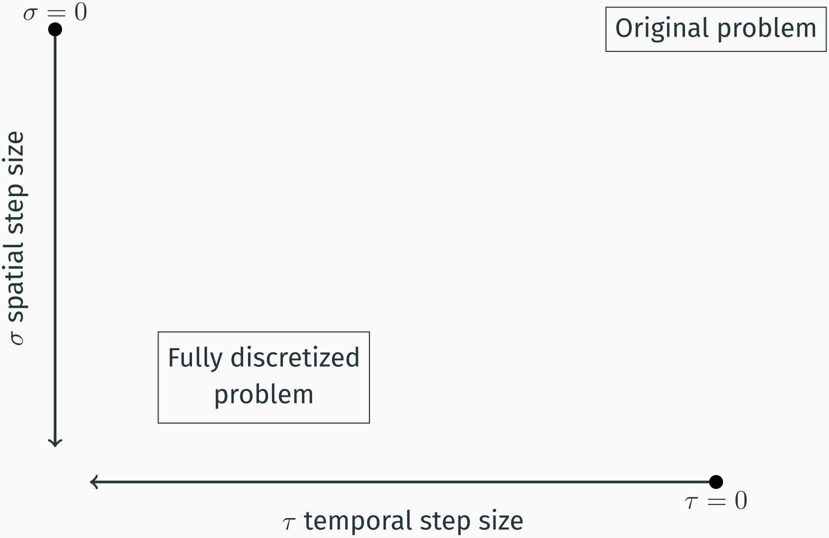
Previous works



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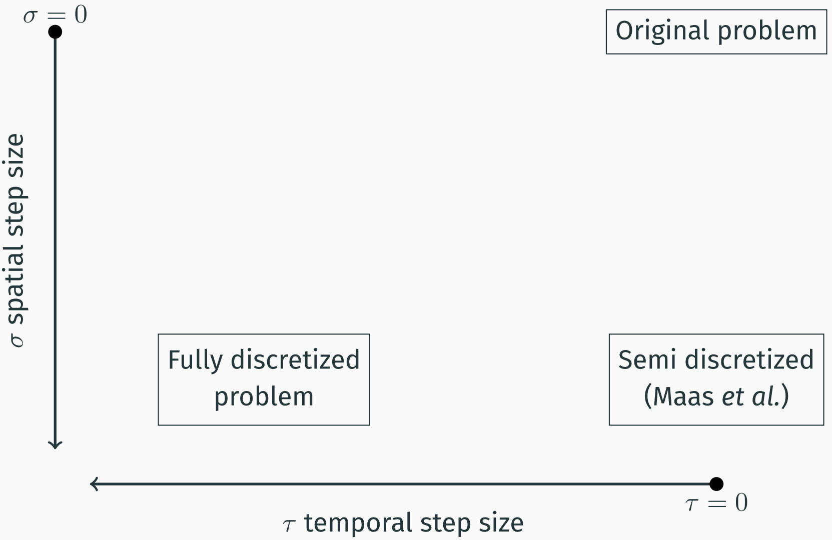

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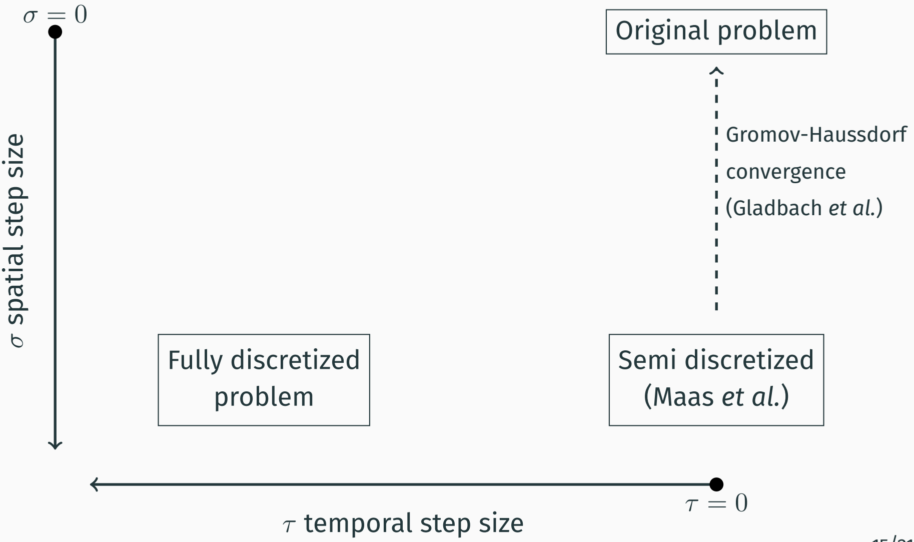

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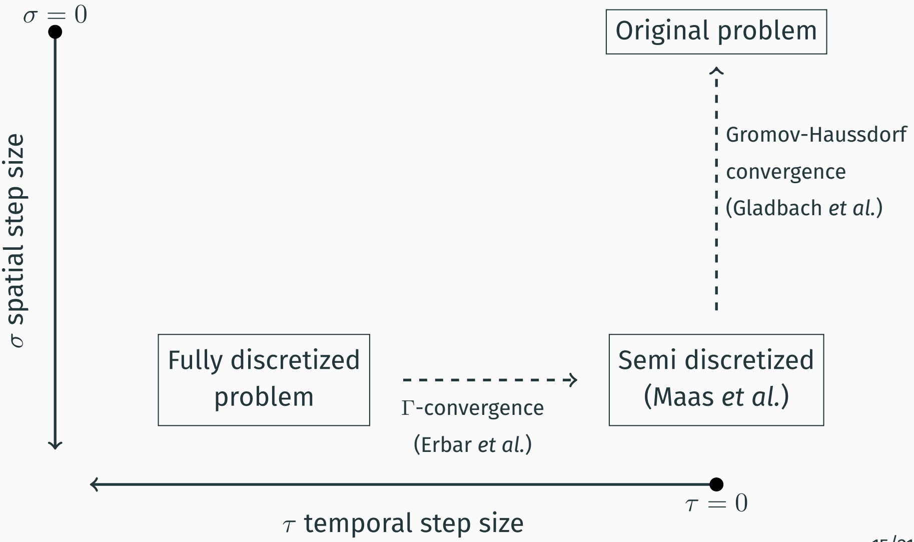

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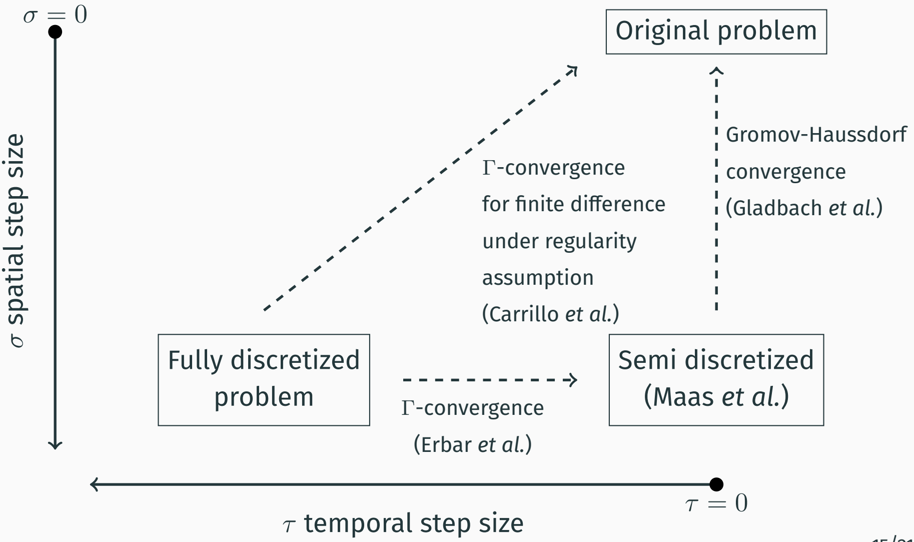

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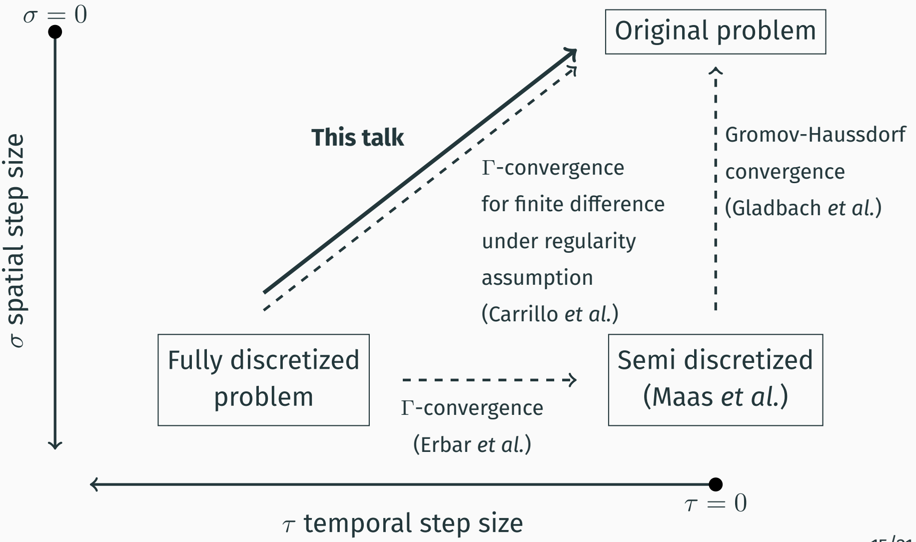

How does the theorem look like?

We assume that X is a smooth Riemannian manifold with a smooth and **convex** boundary.

"Reconstruction" operators $R^{A}_{\mathcal{X}_\sigma}, R^{\mathcal{CE}}_{\mathcal{X}_\sigma} : \mathcal{X}_\sigma \to \mathcal{M}(X)$ and $R_{\mathcal{Y}_\sigma} : \mathcal{Y}_\sigma \to \mathcal{M}(TX)$.

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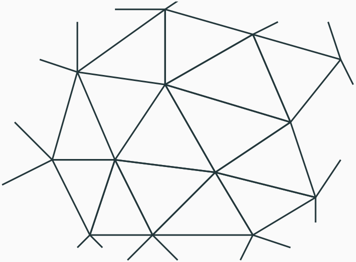
Rough formulation

Under **compatibility conditions** between reconstruction, sampling, A_σ and Div_σ, the solutions of the fully discretized problem, properly reconstructed, **converge weakly in space and time** to a solution of the original problem, when the spatial and temporal grids are refined.

Applications

Triangulations of surfaces



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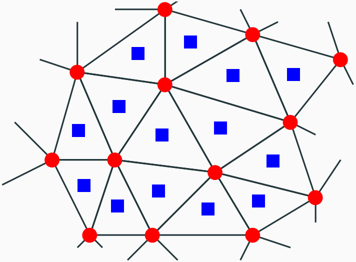

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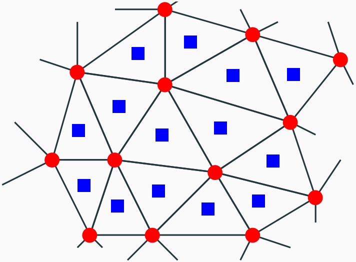

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- Regular meshes.

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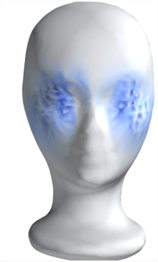

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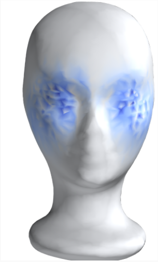

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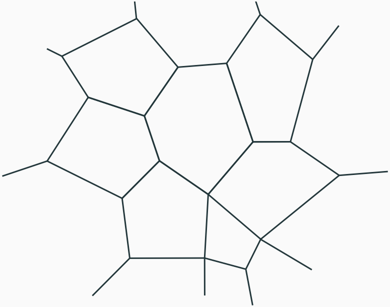

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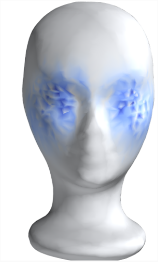

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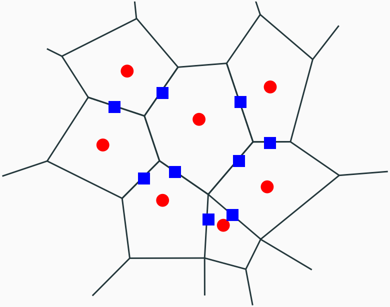

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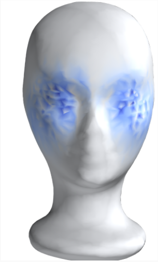

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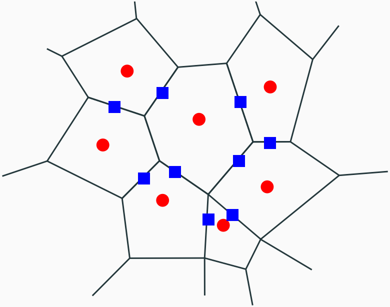

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Works if:

- Admissible, uniformly regular meshes.
- Isotropy condition.

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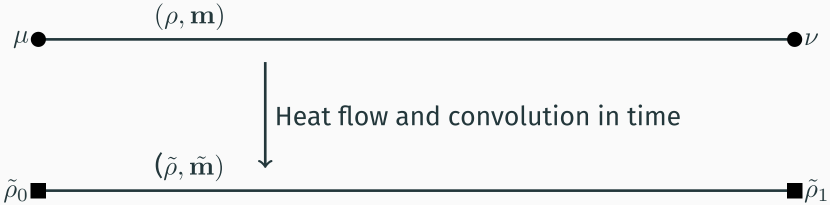

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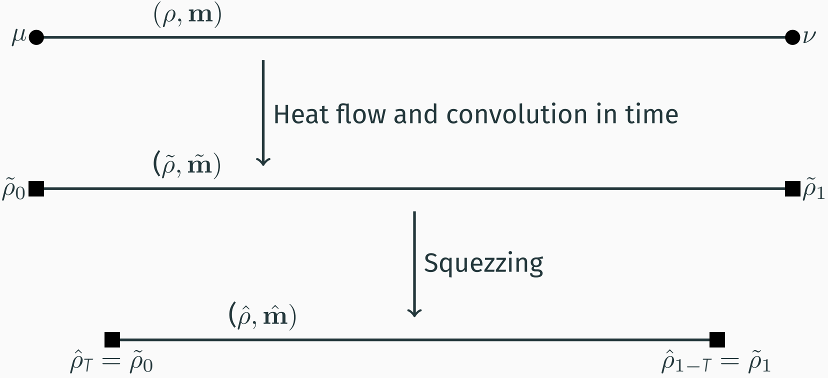

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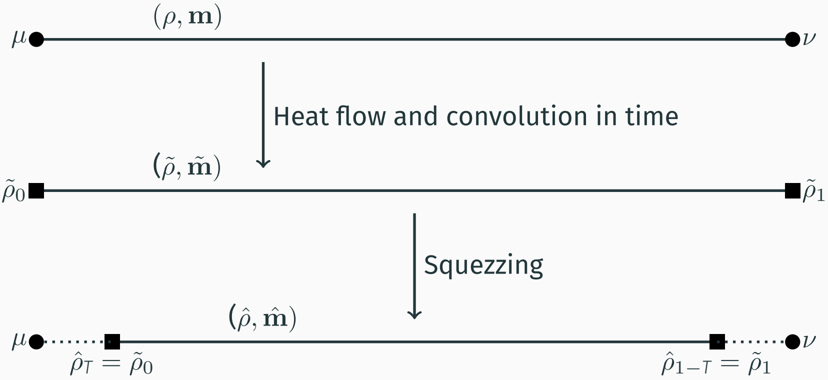

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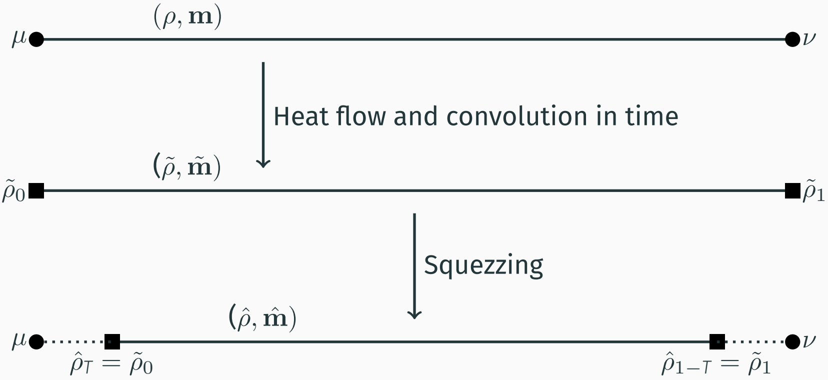
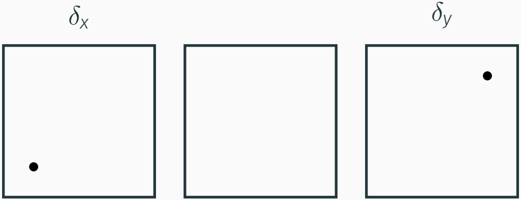


Then sampling the regular part: only consistency is required.

Passing to the limit: controllability

Joining two Dirac masses in one time step with a cost bounded by $d_g(x,y)^2$?



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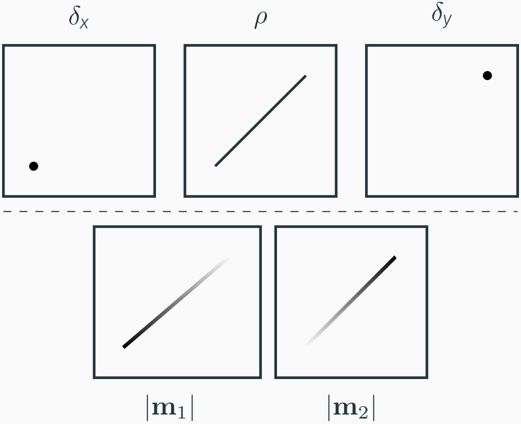

With an appropriate choice of $\mathbf{m}_1, \mathbf{m}_2$,

$$\begin{cases} \nabla \cdot \mathbf{m}_1 = \rho - \delta_x, \\ \nabla \cdot \mathbf{m}_2 = \rho - \delta_y, \end{cases}$$

and

$$\int \frac{|\mathbf{m}_1|^2}{\rho + \delta_x} \lesssim d_g(x,y)^2.$$

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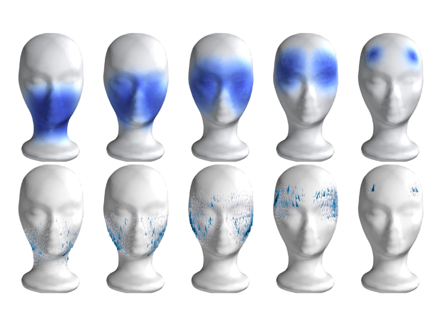

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